6-5 day 2 Logistical Growth

Learning Objectives:

I can use model a population's growth using a logistic growth differential equation.

I can solve a logistic growth differential equation to find an equation that models the population at time t.

I can use the logistic growth equation to find out key information regarding the population growth.

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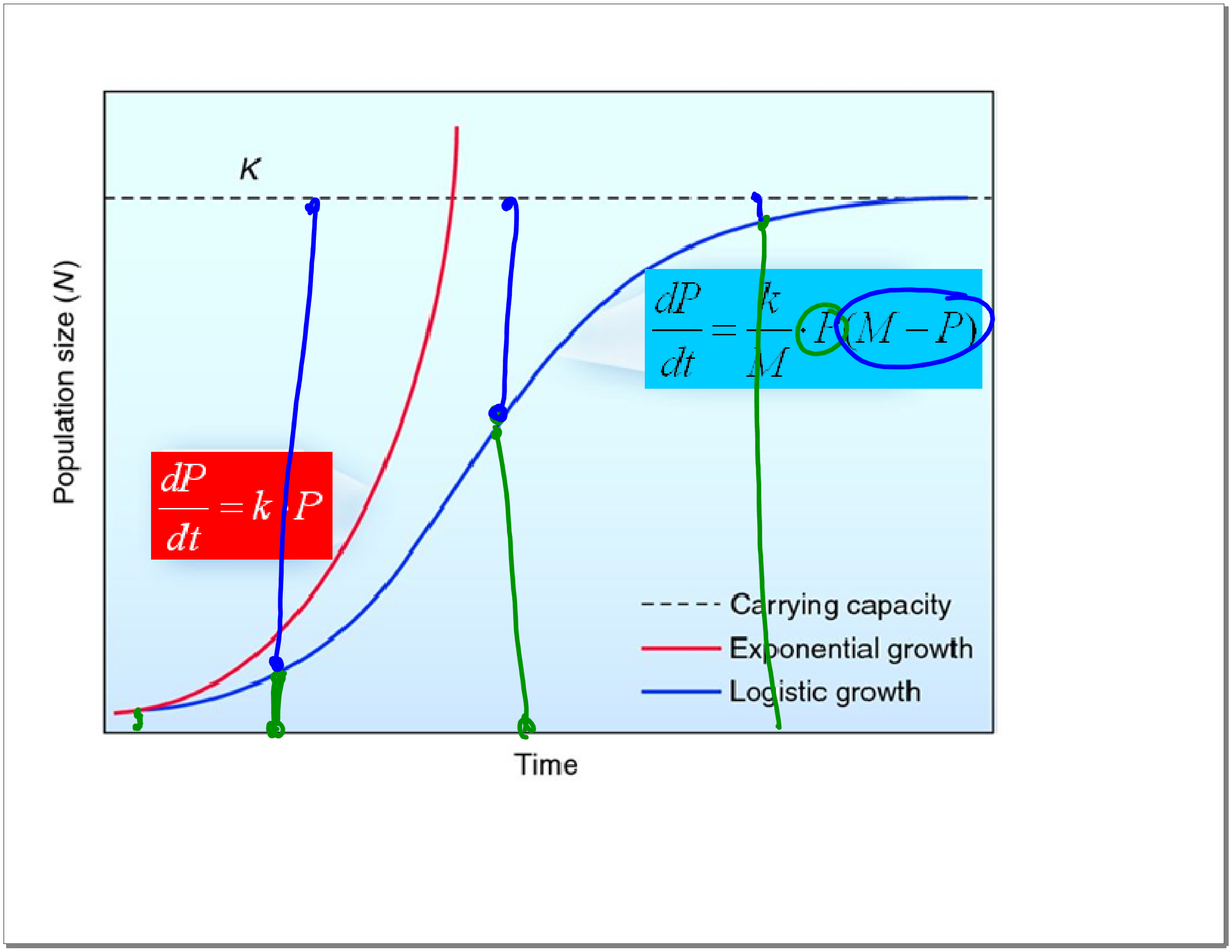
The exponential model for population growth assumes unlimited growth. This is typically not feasible. Usually due to space limitations and limited resources, there is a maximum population (M), called the carrying capacity that the environment is capable of sustaining in the long run. The rate of population growth slows down as the actual population approaches the maximum population.


K
Population size (N)
Time
$\frac{dP}{dt} = k \cdot P$
$\frac{dP}{dt} = \frac{k}{M} \cdot P(M-P)$
Carrying capacity
Exponential growth
Logistic growth

Logistic Growth Differential Equation

A population is growing such that at any given time the growth rate is proportional to both the current population (P) and the difference between the populations carrying capacity and the current population (M-P). This allows the growth rate to approach 0 as the population approaches the carrying capacity. The logistic growth differential equation is thus:

$$\frac{dP}{dt} = \frac{k}{M} \cdot P(M - P)$$

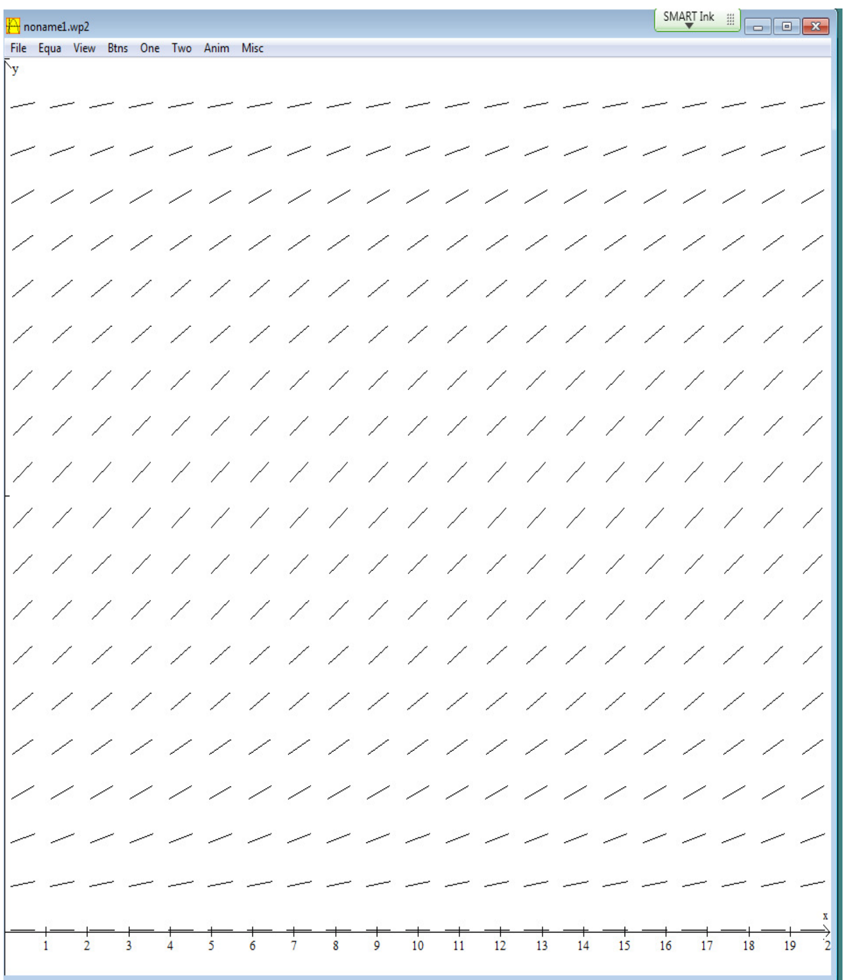
(t, P)

Ex1. A trout pond cannot sustain over 2000 fish. Currently there are 100 trout in the pond and the growth constant is k=.2. $(0, 100)$

a.) Draw the slope field for the logistical growth differential equation

$$\frac{dP}{dt} = \frac{.2}{2000} \cdot P \cdot (2000 - P)$$

$$\frac{dP}{dt} = .0001P(2000-P)$$



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A trout pond cannot sustain over 2000 fish. Currently there are 100 trout in the pond and the growth constant is k=.2.

b.) Solve the differential equation to find an expression for P(t).

$$\frac{dP}{dt} = .0001\,P(2000-P)$$

$$\frac{dP}{P(2000-P)} = .0001\,dt$$

$$\int \frac{1}{P(2000-P)}\,dP = \int .0001\,dt$$

$$\frac{1}{P(2000-P)} = \frac{A}{P} + \frac{B}{2000-P}$$

$$1 = A(2000-P) + B\cdot P$$

$P=0$: $1 = A(2000) \Rightarrow A = \frac{1}{2000}$

$P=2000$: $1 = B\,2000 \Rightarrow B = \frac{1}{2000}$

$$\int \frac{\frac{1}{2000}}{P} + \frac{\frac{1}{2000}}{2000-P}\,dP = \int .0001\,dt$$

$$\frac{1}{2000}\int \frac{1}{P}\,dP + \frac{1}{2000}\int \frac{1}{2000-P}\,dP = \int .0001\,dt$$

$$\frac{1}{2000}\ln P - \frac{1}{2000}\ln(2000-P) = .0001t + C$$

$$\left[\frac{1}{2000}\ln\frac{P}{2000-P} = .0001t + C\right]\cdot 2000$$

$$-\ln\frac{P}{2000-P} = -.2t + C$$

$$\ln\left[\frac{P}{2000-P}\right]^{-1} = -.2t + C$$

$$e^{\ln\frac{2000-P}{P}} = e^{-.2t+C}$$

$$\frac{2000-P}{P} = e^{-.2t+C}$$

$$\frac{2000}{P} - 1 = Ce^{-.2t}$$

$$\frac{2000}{P} = 1 + Ce^{-.2t}$$

$$2000 = (1 + Ce^{-.2t})\cdot P$$

$$\frac{2000}{1+Ce^{-.2t}} = P \quad \text{i.c. } (0,100)$$

$$\frac{2000}{1+C} = 100 \Rightarrow 2000 = 100(1+C)$$

$$20 = 1 + C \Rightarrow C = 19$$

$$\boxed{P = \frac{2000}{1+19e^{-.2t}}}$$

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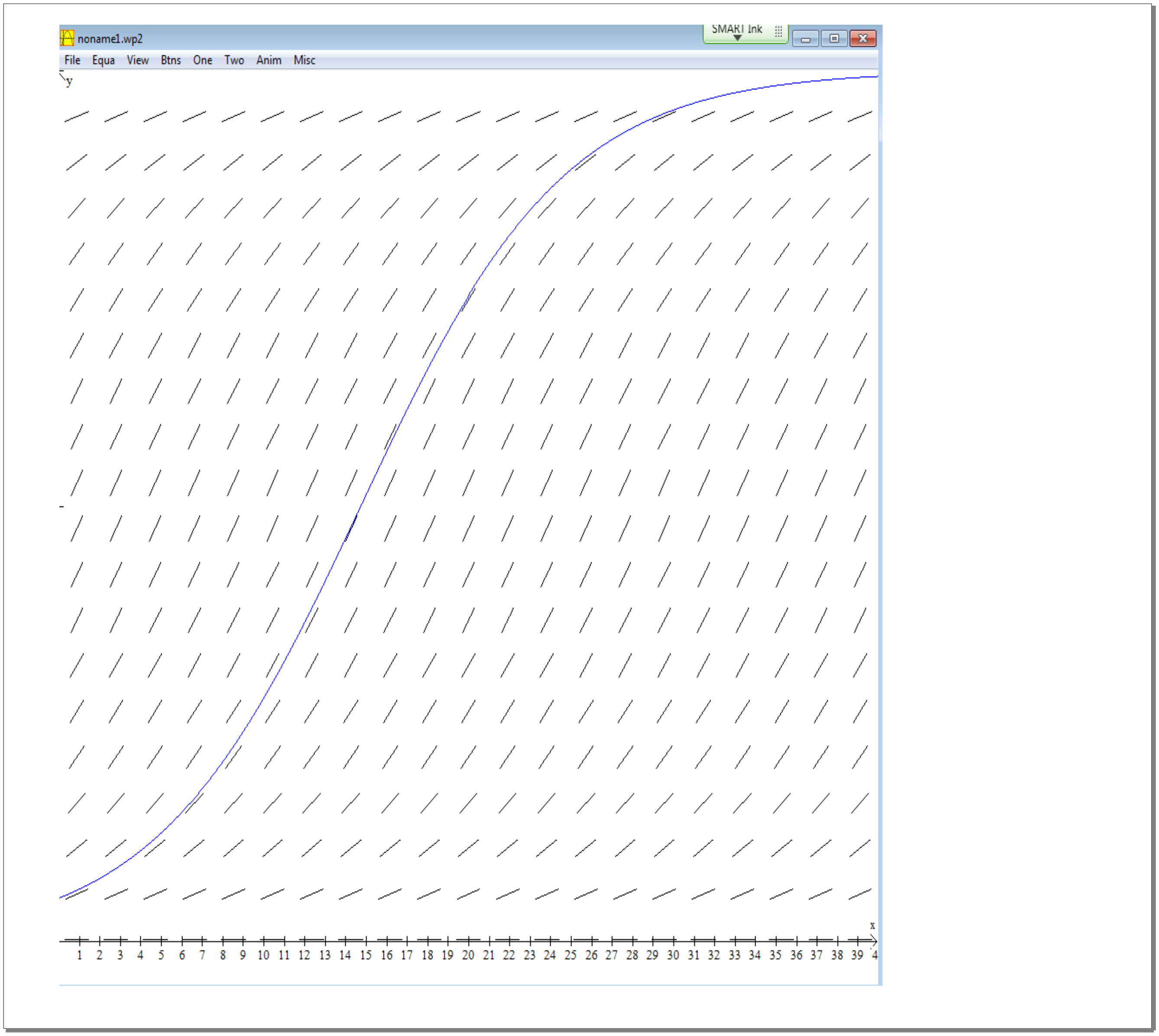
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A trout pond cannot sustain over 2000 fish. Currently there are 100 trout in the pond and the growth constant is k=.2.

c.) When will the trout population reach 1500?

$$P = \frac{2000}{1+19e^{-.2t}}$$

$$1500 = \frac{2000}{1+19e^{-.2t}}$$

$$1500(1+19e^{-.2t}) = 2000$$

$$1+19e^{-.2t} = \frac{4}{3}$$

$$19e^{-.2t} = \frac{1}{3}$$

$$e^{-.2t} = \frac{1}{57}$$

$$-.2t = \ln\left(\frac{1}{57}\right)$$

$$t = \frac{\ln\left(\frac{1}{57}\right)}{-.2} \approx 20.215 \text{ years}$$

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Logistic Growth

$$P = \frac{M}{1 + Ae^{-kt}}$$

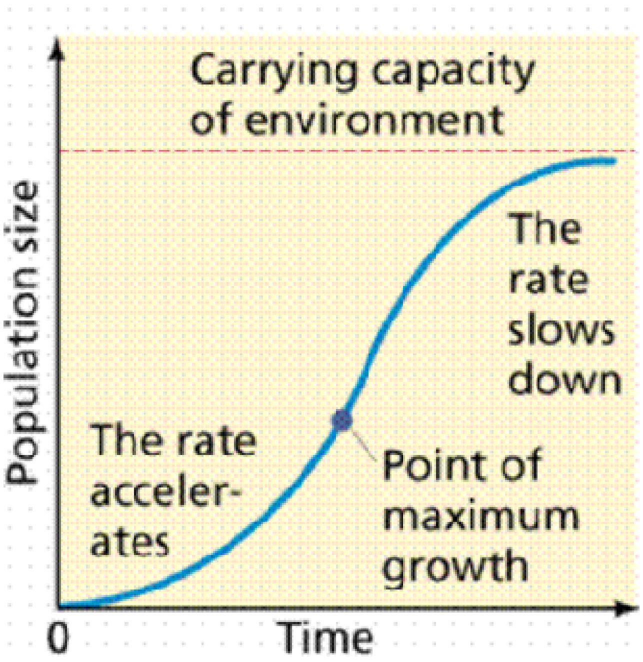
P = Population at time t

t = time

M = Carrying capacity

k = growth constant

A = value determined by some initial condition



In a logistical growth model, the population is growing the fastest when it reaches ½ of the carrying capacity. This point would correspond to the inflection point on the Logistic Growth curve.

Ex3. Suppose a flu-like virus is spreading through a population of 50,000 at a rate proportional both to the number of people already infected and to the number still un-infected. The CDC estimates, due to the highly contagious nature of this virus, that eventually 60% of the population will become infected. If 100 people were infected yesterday and 130 are infected today:

a.) Write the logistic growth differential equation to represent the rate at which the disease is spreading.

b.) Write the logistic growth equation modeling the number of people infected at time t.

b.) At what time t is the number of people infected increasing at the fastest pace? How many people are infected at this time?

c.) Find $\lim_{t\to\infty} P(t)$. Explain what this means in the context of the problem.

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A.)

$P = \frac{30,000}{1+Ae^{-kt}}$ (0, 130) $P = \frac{30000}{1+229.769e^{-k(-1)}}$ (-1, 100)

$130 = \frac{30,000}{1+A}$ $230.769 = 1 + A$ $A = 229.769$

$300 = 1 + 229.769e^{-k(-1)}$ $\ln e^{k} = \ln 1.301$ $k = .263$

$\frac{dP}{dt} = \frac{.263}{30,000} P(30,000 - P)$

B.) Write the logistic growth equation modeling the # of ppl infected at time t

$P = \frac{30,000}{1+229.769e^{-.263t}}$

C.) at 15000 infected.

$15000 = \frac{30000}{1+229.769e^{-.2633t}}$

$15000 + 3446535e^{-.2633t} = 30000$

$3446535e^{-.2633t} = 15000$

$e^{-.2633t} = .00435$

$-.2633t = -5.437$

$t = 20.649$ days

D.) $\lim_{t\to\infty} P(t) = 30,000$, this is the carrying capacity.

Ex4. What is the carrying capacity of the population with growth rate modeled by

$$\frac{dP}{dt} = 6P - .012P^2$$

a.) 500 b.) 50 c.) .012

d.) 6 e.) None of these

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Homework

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