9-3 day 1 Taylor's Theorem

Learning Objectives:

I can use a Taylor polynomial to approximate the value of a function at a given point.

I can use the Lagrange error bound to determine the error associated with using a Taylor (or MacLaurin) polynomial to make an approximation.

Jan 31-2:22 PM

Ex1. Given the function f(x)=cos(x)

a.) Find the 4^{th} degree Taylor polynomial for f(x) centered at x=0.

b.) Use this 4^{th} degree Taylor polynomial to approximate f(1/2)

c.) Find the error associated with using a 4^{th} degree Taylor polynomial to approximate f(1/2).

Jan 31-2:22 PM

Find 4th degree Taylor polynomial for $f(x)$ centered at $x=0$. $f(x)=\cos x$

a) $f(x) \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$

Use 4th degree Taylor polynomial to approx. $f(1/2)$

b) $\cos\left(\frac{1}{2}\right) \approx 1 - \frac{(1/2)^2}{2!} + \frac{(1/2)^4}{4!} = 1 - \frac{\frac{1}{4}}{2} + \frac{\frac{1}{16}}{24}$

$1 - \frac{1}{8} + \frac{1}{384} = \frac{384}{384} - \frac{48}{384} + \frac{1}{384}$

$= \boxed{\frac{337}{384}}$

Find error using 4th degree Taylor poly to approx. $f(1/2)$

c) $\text{error} < \frac{x^6}{6!} \longrightarrow \frac{\left(\frac{1}{2}\right)^6}{6!}$ $\quad \text{error} < \frac{1}{46{,}080}$

Mar 11-12:33 PM

Ex2.

a.) Write the terms for the Taylor series for f(x)=ln(x) centered at x=1.

b.) Use the 3^{rd} degree Taylor Polynomial to approximate the value of f(3/2)

b.) Determine the error associated with making this approximation.

c.) Find the exact error.

Feb 28-4:28 PM

Ex1. a.) Find the 3rd degree Taylor Polynomial for $\ln x$ centered at $a = 1$

$f(x) = \ln x$ $\quad f(1) = 0$

$f'(x) = 1/x$ $\quad f'(1) = 1$

$f''(x) = -1/x^2$ $\quad f''(1) = -1$

$f'''(x) = 2/x^3$ $\quad f'''(1) = 2$

$$P_3(x) = \frac{0}{0!} + \frac{1}{1!}(x-1) + \frac{-1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3$$

$$\boxed{P_3(x) = (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3}$$

b.) Use this 3rd degree Taylor Polynomial to approximate $f(1.2)$

$$P_3(1.2) = (1.2-1) - \tfrac{1}{2}(1.2-1)^2 + \tfrac{1}{3}(1.2-1)^3$$
$$= .2 - \tfrac{1}{2}(.2)^2 + \tfrac{1}{3}(.2)^3$$
$$= .2 - \tfrac{1}{2}(.04) + \tfrac{1}{3}(.008)$$
$$= .2 - .02 + .00267 \approx \boxed{.18267}$$

c.) Use the Lagrange Form of Taylors Theorem to find out how "good" this approximation is.

$$R_n(x) \; \frac{f^{n+1}(c)}{(n+1)!}(b-a)^{n+1}$$

$$R_3(1.2) = \frac{f^{n+1}(c)}{4!}(1.2-1)^4$$

this is really hard to find

$$R_3(1.2) = \frac{K\,(.2)^4}{4!}$$

$f^{IV}(x) = -6x^{-4} = \frac{-6}{x^4}$

↑ this is a decreasing fcn so it's largest value will occur at $x = 1$

$|f^{IV}(1)| = 6$

$$R_3(1.2) = \frac{6 \cdot .0016}{24} = \tfrac{1}{4}(.0016) = .0004$$

$$R_3(1.2) \le .0004$$

d.) Find the exact error

$$\ln 1.2 - .18267 = .0003484 \le .0004$$

Mar 11-12:30 PM

Ex3. Given the function $f(x) = e^x$

a.) Find the 3^{rd} degree Taylor polynomial for f(x) centered at x=0.

b.) Use this 3^{rd} degree Taylor polynomial to approximate f(1/2)

c.) Find the error associated with using a 3^{rd} degree Taylor polynomial to approximate f(1/2).

Jan 31-2:22 PM

$f(x) = e^x$. 3^{rd} degree Taylor Polynomial, Center $x = 0$

a) $f(x) \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

b) $f(1/2) \approx 1 + \frac{1}{2} + \frac{(\frac{1}{2})^2}{2!} + \frac{(\frac{1}{2})^3}{3!} = 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{48} = \frac{79}{48}$

c) $a = 0$

$b = \frac{1}{2}$

$n = 3$

$n+1 = 4$

$0 \le c \le \frac{1}{2}$

0 c 1/2

$\text{error} = \frac{f^{4}(c)\,(\frac{1}{2} - 0)^4}{4!}$

$$\boxed{\text{error} \le \frac{\boxed{e^{1/2}}\,(\frac{1}{2})^4}{4!}}$$

$f'''' = e^x$ $f''''(c) \le e^{1/2}$

$f(x) = P_n(x) + R_n(x)$

actual function

n^{th} degree Taylor approx

error when you approx n^{th} degree Taylor polynomial

Our next goal is to study the error when we use an nth degree Taylor Polynomial to approximate the value of a function. We will use define the error as the remainder.

$$f(x) = P_n(x) + R_n(x)$$

Actual Function

Error when you approx with the n^{th} degree Taylor polynomial

n^{th} degree Taylor approx

Jan 31-2:22 PM

Taylor's Theorem

If f(x) has derivatives of all orders and we use an nth degree Taylor polynomial centered at x=a to approximate f(b) as such

$$f(x)=f(a)+f'(a)(b-a)+\frac{f''(a)}{2!}(b-a)^2+\frac{f'''(a)}{3!}(b-a)^3\ldots+\frac{f^n(a)}{n!}(b-a)^n$$

then the remainder is given by $R_n(x)=\frac{f^{n+1}(c)}{(n+1)!}(b-a)^{n+1}$

for some c between $a \le c \le b$

Jan 31-2:22 PM

Lagrange Error Bound

Let k be some number such that $\left|f^{n+1}(c)\right| \leq k$

for all c in $a \leq c \leq b$, then the error is given by

$$R_n(x) \leq k \cdot \frac{|b-a|^{n+1}}{(n+1)!}$$

Jan 31-2:22 PM

Ex4.

a.) Write the terms for the Mclaurin series for $f(x)=xe^x$

b.) Use the 4th degree McLaurin Polynomial to approximate the value of f(1/4)

Write McLaurin Series for xe^x.

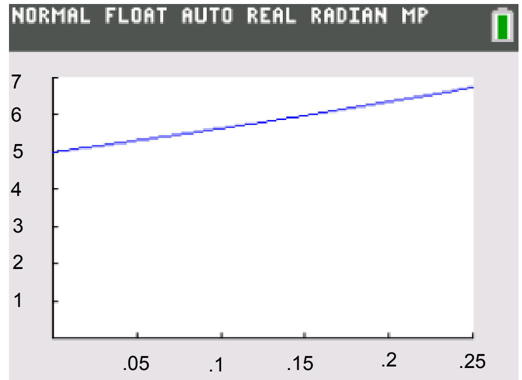
a) $f(x) = xe^x$

$e^x \rightarrow 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

$xe^x \rightarrow x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!}$

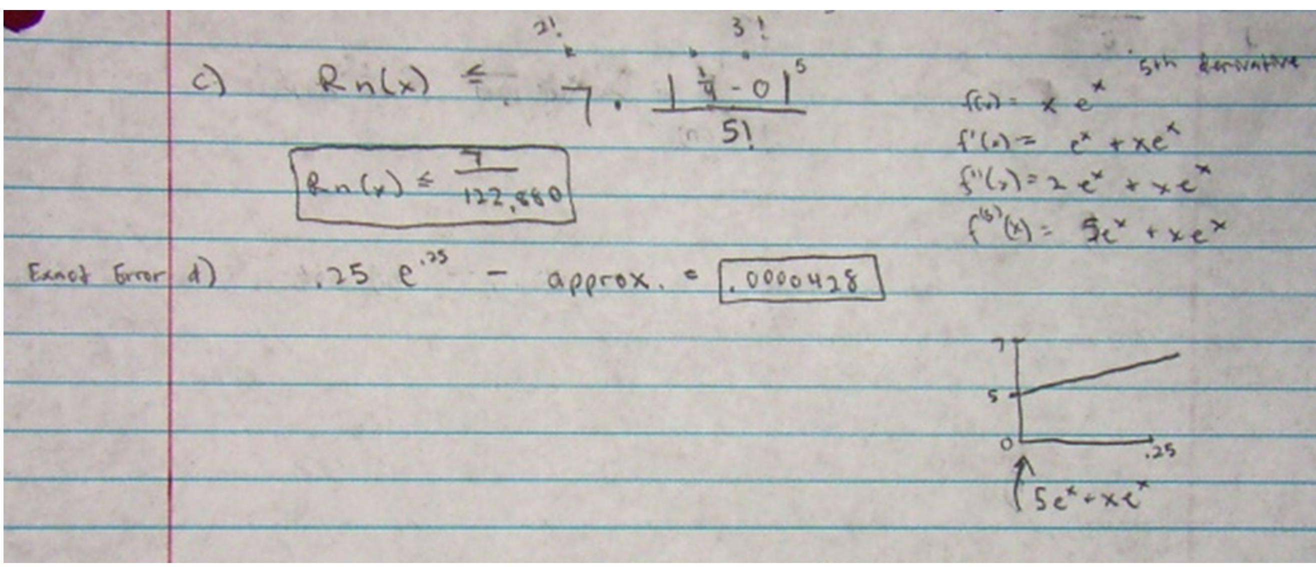
b) $\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \frac{\left(\frac{1}{4}\right)^3}{2!} + \frac{\left(\frac{1}{4}\right)^4}{3!} = .32096$

The graph of $y = 5e^x + xe^x$ is provided



c.) Use the Lagrange Error form of Taylor's Theorem to find out how "good" this approximation is.

d.) Find the exact error.



Homework

Pg 500 #1-5 (and find error), 22, 23, 33, 39-41, 43

Jan 31-2:22 PM