3 Algebraic Functions, Equations and Inequalities

Assessment statements

2.1 Odd and even functions (also see Chapter 7).

2.4 The rational function $x \mapsto \dfrac{ax+b}{cx+d}$ and its graph.

2.5 Polynomial functions.
The factor and remainder theorems.
The fundamental theorem of algebra.

2.6 The quadratic function $x \mapsto ax^2 + bx + c$: its graph, axis of symmetry $x = -\dfrac{b}{2a}$.
The solution of $ax^2 + bx + c = 0$, $a \neq 0$.
The quadratic formula. Use of the discriminant $\Delta = b^2 - 4ac$.
Solving equations both graphically and algebraically.
Sum and product of the roots of polynomial equations.

2.7 Solution of inequalities $g(x) \geqslant f(x)$; graphical and algebraic methods.

Introduction

A function $x \mapsto f(x)$ is called **algebraic** if, substituting for the number x in the domain, the corresponding number $f(x)$ in the range can be computed using a finite number of **elementary operations** (i.e. addition, subtraction, multiplication, division, and extracting a root). For example, $f(x) = \dfrac{x^2 + \sqrt{9 - x}}{2x - 6}$ is algebraic. For our purposes in this course, functions can be organized into three categories:

1. Algebraic functions
2. Exponential and logarithmic functions (Chapter 5)
3. Trigonometric and inverse trigonometric functions (Chapter 7)

The focus of this chapter is algebraic functions of a single variable which – given the definition above – are functions that contain polynomials, radicals (surds), rational expressions (quotients), or a combination of these. The chapter will begin by looking at polynomial functions in general and then moves onto a closer look at 2nd degree polynomial functions (quadratic functions). Solving equations containing polynomial functions is an important skill that will be covered. We will also study rational functions, which are quotients of polynomial functions and the associated topic of partial fractions (optional). The chapter will close with methods of solving inequalities and absolute value functions, and strategies for solving various equations.

The concept of a function is a fairly recent development in the history of mathematics. Its meaning started to gain some clarity about the time of René Descartes (1596–1650) when he defined a function to be any positive integral power of x (i.e. x^2, x^3, x^4, etc.). Leibniz (1646–1716) and Johann Bernoulli (1667–1748) developed the concept further. It was Euler (1707–1783) who introduced the now standard function notation $y = f(x)$.

3.1 Polynomial functions

The most common type of algebraic function is a polynomial function where, not surprisingly, the function's rule is given by a polynomial. For example,

$$f(x) = x^3, \quad h(t) = -2t^2 + 16t - 24, \quad g(y) = y^5 + y^4 - 11y^3 + 7y^2 + 10y - 8$$

Recalling the definition of a polynomial, we define a polynomial function.

Definition of a polynomial function in the variable x

A **polynomial function** P is a function that can be expressed as

$$P(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0, \quad a_n \neq 0$$

where the non-negative integer n is the **degree** of the polynomial function. The numbers $a_0, a_1, a_2, \dots, a_n$, are real numbers and are the **coefficients** of the polynomial. a_n is the **leading coefficient**, a_nx^n is the **leading term** and a_0 is the **constant term**.

It is common practice to use subscript notation for coefficients of general polynomial functions, but for polynomial functions of low degree, the following simpler forms are often used.

Degree	Function form	Function name	Graph
Zero	$P(x) = a$	Constant function	Horizontal line
First	$P(x) = ax + b$	Linear function	Line with slope a
Second	$P(x) = ax^2 + bx + c$	Quadratic function	Parabola (∪-shape, 1 turn)
Third	$P(x) = ax^3 + bx^2 + cx + d$	Cubic function	$\cap\!\cup$-shape (2 or no turns)

Table 3.1 Features of polynomial functions of low degree.

To identify an individual term in a polynomial function, we use the function name correlated with the power of x contained in the term. For example, the polynomial function $f(x) = x^3 - 9x + 4$ has a *cubic* term of x^3, no *quadratic* term, a *linear* term of $-9x$, and a *constant* term of 4.

For each polynomial function $P(x)$ there is a corresponding **polynomial equation** $P(x) = 0$. When we solve polynomial equations, we often refer to solutions as **roots**.

• **Hint:** When working with a polynomial function, such as $f(x) = x^3 - 9x + 4$, it is common to refer to it in a couple of different ways – either as 'the polynomial $f(x)$', or as 'the function $x^3 - 9x + 4$.'

Zeros and roots

If P is a function and c is a number such that $P(c) = 0$, then c is a **zero** of the function P (or of the polynomial P) and $x = c$ is a **root** of the equation $P(x) = 0$.

● **Hint:** The use of the word '**root**' here to denote the solution of a polynomial equation should not be confused with the use of the word in the context of square root, cube root, fifth root, etc.

Approaches to finding zeros of various polynomial functions will be considered in the first three sections of this chapter.

Graphs of polynomial functions

As we reviewed in Section 1.6, the graph of a first-degree polynomial function (linear function), such as $P(x) = 2x - 5$, is a line (Figure 3.1a). The graph of every second-degree polynomial function (quadratic function) is a parabola (Figure 3.1b). A thorough review and discussion of quadratic functions and their graphs is in the next section.

The simplest type of polynomial function is one whose rule is given by a power of x. In Figure 3.1, the graphs of $P(x) = x^n$ for $n = 1, 2, 3, 4, 5$ and 6 are shown. As the figure suggests, the graph of $P(x) = x^n$ has the same general ∪-shape as $y = x^2$ when n is even, and the same general ∩∪ shape as $y = x^3$ when n is odd. However, as the degree n increases, the graphs of polynomial functions become flatter near the origin and steeper away from the origin.

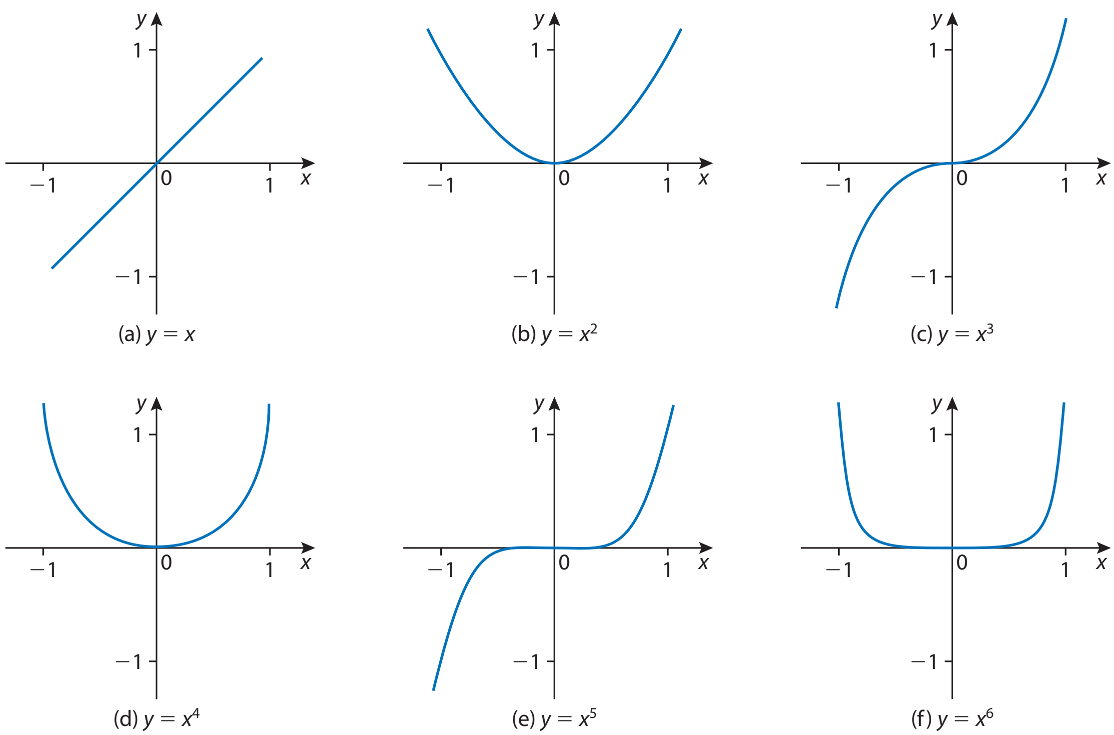
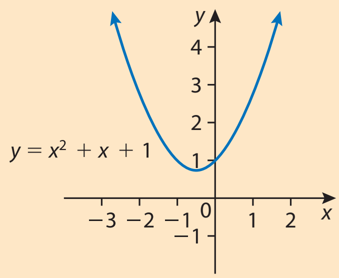


Figure 3.1 Graphs of $P(x) = x^n$ for increasing n.

Another interesting observation is that, depending on the degree of the polynomial function, its graph displays a certain type of symmetry. The graph of $P(x) = x^n$ is symmetric with respect to the origin when n is odd. Such a function is aptly called an **odd function**. The graph of $P(x) = x^n$ is symmetric with respect to the y-axis when n is even. Accordingly any such function is called an **even function**. Formal definitions for odd and even functions will be presented in Chapter 7 when we investigate the graphs of the sine and cosine functions.

Not all polynomial functions are even or odd – that is, not all polynomial functions display rotation symmetry about the origin or reflection symmetry about the y-axis. For example, the graph of the polynomial function $y = x^2 + x + 1$ is neither even nor odd. It has line symmetry, but the line of symmetry is not the y-axis.



Note that the graph of an **even function** may or may not intersect the x-axis (x-intercept). As we will see, where and how often the graph of a function intersects the x-axis is helpful information when trying to determine the value and nature of the roots of a polynomial equation $P(x) = 0$.

The graphs of polynomial functions that are not in the form $P(x) = x^n$ are more difficult to sketch. However, the graphs of all polynomial functions share these properties:

1. It is a smooth curve (i.e. it has no sharp, pointed turns – only smooth, rounded turns).
2. It is continuous (i.e. it has no breaks, gaps or holes).
3. It rises ($P(x) \to \infty$) or falls ($P(x) \to -\infty$) without bound as $x \to +\infty$ or $x \to -\infty$.
4. It extends on forever both to the left ($-\infty$) and to the right ($+\infty$); domain is $\mathbb{R}$.
5. The graph of a polynomial function of degree n has at most $n - 1$ turning points.

The property that is listed third of the five properties of the graphs of polynomial functions is referred to as the **end behaviour** of the function because it describes how the curve *behaves* at the left and right *ends* (i.e. as $x \to +\infty$ and as $x \to -\infty$). The end behaviour of a polynomial function is determined by its degree and by the sign of its leading coefficient. See Exercise 3.1, Q11.

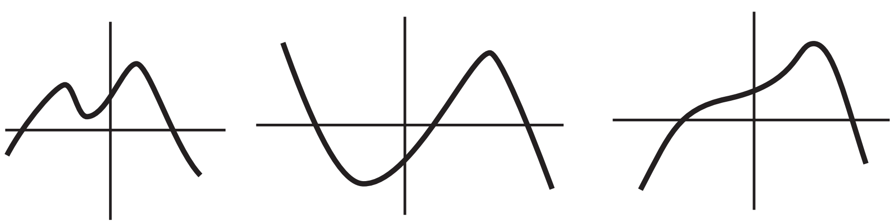


Figure 3.2 The graph of a polynomial function is a smooth, unbroken, continuous curve, such as the ones shown here.

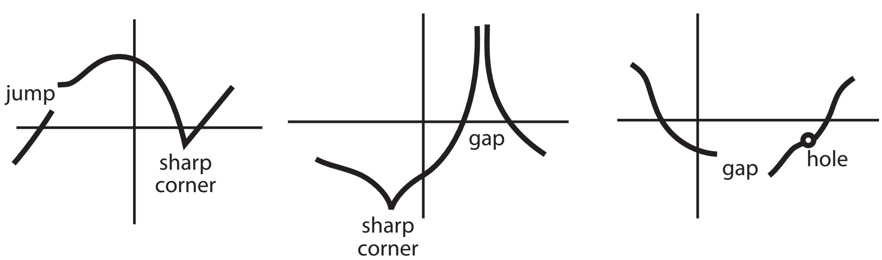


Figure 3.3 There can be no jumps, gaps, holes or sharp corners on the graph of a polynomial function. Thus none of the functions whose graphs are shown here are polynomial functions.

If we wish to sketch the graph of a polynomial function without a GDC, we need to compute some function values in order to locate a few points on the graph. This could prove to be quite tedious if the polynomial function has a high degree. We will now develop a method that provides an efficient procedure for evaluating polynomial functions. It will also be useful in the third section of this chapter for some situations when we divide polynomials. For simplicity, we give the method for a fourth-degree polynomial, but it is applicable to any nth degree polynomial.

Synthetic substitution (Optional)

Suppose we want to find the value of $P(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ when $x = c$, that is, find $P(c)$. The computation of c^4 may be tricky, so rather than substituting c directly into $P(x)$ we will take a gradual approach that consists of a sequence of multiplications and additions. We define b_4, b_3, b_2, b_1, and R by the following equations.

$$b_4 = a_4 \quad (1)$$

$$b_3 = b_4c + a_3 \quad (2)$$

$$b_2 = b_3c + a_2 \quad (3)$$

$$b_1 = b_2c + a_1 \quad (4)$$

$$R = b_1c + a_0 \quad (5)$$

Our goal is to show that the value of $P(c)$ is equivalent to the value of R. Firstly, we substitute the expression for b_3 given by equation (2) into equation (3), and also use equation (1) to replace b_4 with a_4, to produce

$$\begin{aligned} b_2 &= (a_4c + a_3)c + a_2 \\ &= a_4c^2 + a_3c + a_2 \qquad (6) \end{aligned}$$

We now substitute this expression for b_2 in (6) into (4) to give

$$\begin{aligned} b_1 &= (a_4c^2 + a_3c + a_2)c + a_1 \\ &= a_4c^3 + a_3c^2 + a_2c + a_1 \qquad (7) \end{aligned}$$

To complete our goal we substitute this expression for b_1 in (7) into (5) to give

$$\begin{aligned} R &= (a_4c^3 + a_3c^2 + a_2c + a_1)c + a_0 \\ &= a_4c^4 + a_3c^3 + a_2c^2 + a_1c + a_0 \qquad (8) \end{aligned}$$

This is the value of $P(x)$ when $x = c$. If we condense (6), (7) and (8) into one expression, we obtain

$$\begin{aligned} R &= \{[(a_4c + a_3)c + a_2]c + a_1\}c + a_0 \\ &= a_4c^4 + a_3c^3 + a_2c^2 + a_1c + a_0 = P(c) \qquad (9) \end{aligned}$$

Carrying out the computations for equation (9) can be challenging. However, a nice pattern can be found if we closely inspect the expression $\{[(a_4c + a_3)c + a_2]c + a_1\}c + a_0$. Each nested computation involves finding the product of c and one of the coefficients, a_n, (starting with the leading coefficient) and then adding the next coefficient – and repeating this process until the constant term is used. Hence, the actual computation of R is quite straightforward if we arrange the nested computations required for (9) in the following systematic manner.

$$\begin{array}{r|ccccc}
c & a_4 & a_3 & a_2 & a_1 & a_0 \\
 & & c \times b_4 & c \times b_3 & c \times b_2 & c \times b_1 \\
\hline
 & b_4 & b_3 & b_2 & b_1 & R \quad = P(c)
\end{array}$$

In this procedure we place c in a small box to the upper left. The coefficients of the polynomial function $P(x)$ are placed in the first line. We start by simply rewriting the leading coefficient below the horizontal line (remember $b_4 = a_4$). The diagonal arrows indicate that we multiply the number in the row below the line by c to obtain the next number in the second row above the line. Each b_n after the leading coefficient is obtained by adding the two numbers in the first and second rows directly above b_n. At the end of the procedure, the last such sum is $R = P(c)$. This method of computing the value of $P(x)$ when $x = c$ is called **synthetic substitution**.

Example 1 – Using synthetic substitution to find function values

Given $P(x) = 2x^4 + 6x^3 - 5x^2 + 7x - 12$, find the value of $P(x)$ when $x = -4, -1$ and 2.

Solution

We use the procedure for synthetic substitution just described.

$$\begin{array}{r|ccccc}
-4 & 2 & 6 & -5 & 7 & -12 \\
 & & -8 & 8 & -12 & 20 \\
\hline
 & 2 & -2 & 3 & -5 & 8 \quad = P(-4)
\end{array}$$

Therefore, $P(-4) = 8$.

Note: Contrast using synthetic substitution to evaluate $P(-4)$ with using direct substitution.

$$\begin{aligned}
P(-4) &= 2(-4)^4 + 6(-4)^3 - 5(-4)^2 + 7(-4) - 12 \\
&= 2(256) + 6(-64) - 5(16) - 28 - 12 \\
&= 512 - 384 - 80 - 28 - 12 \\
&= 128 - 108 - 12 \\
&= 8
\end{aligned}$$

$$\begin{array}{r|ccccc}
-1 & 2 & 6 & -5 & 7 & -12 \\
 & & -2 & -4 & 9 & -16 \\
\hline
 & 2 & 4 & -9 & 16 & -28 \quad = P(-1)
\end{array}$$

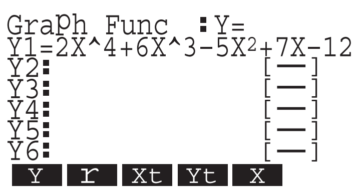
Therefore, $P(-1) = -28$.

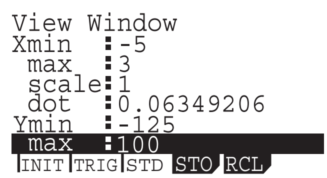
$$\begin{array}{c|ccccc}
2 & 2 & 6 & -5 & 7 & -12 \\
 & & 4 & 20 & 30 & 74 \\
\hline
 & 2 & 10 & 15 & 37 & 62 \end{array} = P(2)$$

Therefore, $P(2) = 62$.

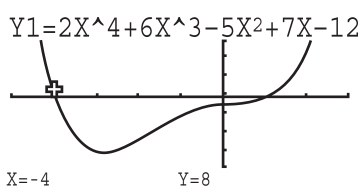
Since the graphs of all polynomial functions are continuous (no gaps or holes), then the function values we computed for the quartic polynomial function in Example 1 can give us information about the location of its zeros (i.e. x-intercepts of the graph). Since $P(-4) = 8$ and $P(-1) = -28$, then the graph of $P(x)$ must cross the x-axis ($P(x) = 0$) at least once between $x = -4$ and $x = -1$. Also, with $P(-1) = -28$ and $P(2) = 62$ there must be at least one x-intercept between $x = -1$ and $x = 2$. Hence, the polynomial equation $P(x) = 2x^4 + 6x^3 - 5x^2 + 7x - 12 = 0$ has at least one real root between -4 and -1, and at least one real root between -1 and 2. In Section 3.3 we will investigate real zeros of polynomial functions and then we will extend the investigation to include imaginary zeros, thereby extending the universal set for solving polynomial equations from the real numbers to complex numbers.

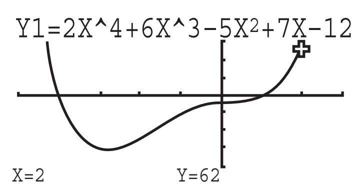
Graphing $P(x) = 2x^4 + 6x^3 - 5x^2 + 7x - 12$ on our GDC, we observe that the graph of $P(x)$ does indeed intersect the x-axis between -4 and -1 (just slightly greater than $x = -4$), and again between -1 and 2 (near $x = 1$).





● **Hint:** For some values of x, evaluating $P(x)$ by direct substitution may be quicker than using synthetic substitution. This is certainly true when $x = 0$ or $x = 1$. For example, it is easy to determine that $P(0) = -12$ for the polynomial P in Example 1; and that $P(1) = 2 + 6 - 5 + 7 - 12 = -2$.





Example 2

Use synthetic substitution to find the y-coordinates of the points on the graph of $f(x) = x^3 - 4x^2 + 24$ for $x = -3, -1, 1, 3$ and 5. Sketch the graph of f for $-4 \leqslant x \leqslant 6$.

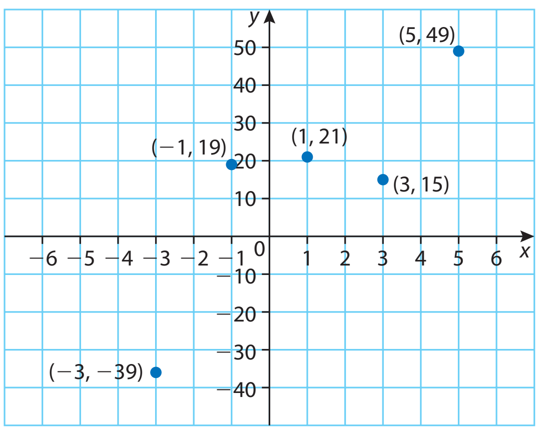
Solution

Important: In order for the method of synthetic substitution to work properly it is necessary to insert 0 for any 'missing' terms in the polynomial. The polynomial $x^3 - 4x^2 + 24$ has no linear term so the top row in the set-up for synthetic substitution must be 1 -4 0 24.

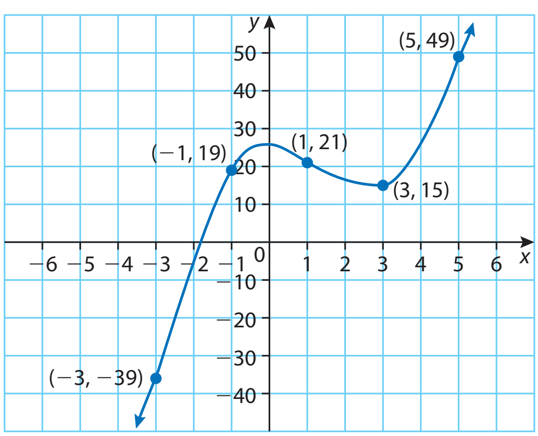
$$\begin{array}{r|rrrr} -3 & 1 & -4 & 0 & 24 \\ & & -3 & 21 & -63 \\ \hline & 1 & -7 & 21 & -39 \end{array} \qquad \begin{array}{r|rrrr} -1 & 1 & -4 & 0 & 24 \\ & & -1 & 5 & -5 \\ \hline & 1 & -5 & 5 & 19 \end{array} \qquad \begin{array}{r|rrrr} 1 & 1 & -4 & 0 & 24 \\ & & 1 & -3 & -3 \\ \hline & 1 & -3 & -3 & 21 \end{array}$$

$$\begin{array}{r|rrrr} 3 & 1 & -4 & 0 & 24 \\ & & 3 & -3 & -9 \\ \hline & 1 & -1 & -3 & 15 \end{array} \qquad \begin{array}{r|rrrr} 5 & 1 & -4 & 0 & 24 \\ & & 5 & 5 & 25 \\ \hline & 1 & 1 & 5 & 49 \end{array}$$

Therefore, the points $(-3, -39)$, $(-1, 19)$, $(1, 21)$, $(3, 15)$ and $(5, 49)$ are on the graph of f and have been plotted in the coordinate plane below.



Recall that the end behaviour of a polynomial function is determined by its degree and by the sign of its leading coefficient. Since the leading term of f is x^3 then its graph will fall ($y \to -\infty$) as $x \to -\infty$ and will rise ($y \to \infty$) as $x \to +\infty$. Also a polynomial function of degree n has at most $n - 1$ turning points; therefore, the graph of f has at most two turning points. Given the coordinates of the five points found with the aid of synthetic substitution, there will clearly be exactly two turning points. The graph of f can now be accurately sketched.



Exercise 3.1

In questions 1–4, use synthetic substitution to evaluate $P(x)$ for the given values of x.

1 $P(x) = x^4 + 2x^3 - 3x^2 - 4x - 20, \quad x = 2, \; x = -3$

2 $P(x) = 2x^5 - x^4 + 3x^3 - 15x - 9, \quad x = -1, \; x = 2$

3 $P(x) = x^5 + 5x^4 + 3x^3 - 6x^2 - 9x + 11, \quad x = -2, \; x = 4$

4 $P(x) = x^3 - (c + 3)x^2 + (3c + 5)x - 5c, \quad x = c, \; x = 2$

5 Given $P(x) = kx^3 + 2x^2 - 10x + 3$, for what value of k is $P(-2) = 15$?

6 Given $P(x) = 3x^4 - 2x^3 - 10x^2 + 3kx + 3$, for what value of k is $x = -\frac{1}{3}$ a zero of $P(x)$?

For questions 7 and 8, do not use your GDC.

7 a) Given $y = 2x^3 + 3x^2 - 5x - 4$, determine the y-value for each value of x such that $x \in \{-3, -2, -1, 0, 1, 2, 3\}$.

b) How many times must the graph of $y = 2x^3 + 3x^2 - 5x - 4$ cross the x-axis?

c) Sketch the graph of $y = 2x^3 + 3x^2 - 5x - 4$.

8 a) Given $y = x^4 - 4x^2 - 2x + 1$, determine the y-value for each value of x such that $x \in \{-3, -2, -1, 0, 1, 2, 3\}$.

b) How many times must the graph of $y = x^4 - 4x^2 - 2x + 1$ cross the x-axis?

c) Sketch the graph of $y = x^4 - 4x^2 - 2x + 1$.

9 Given $f(x) = x^3 + ax^2 - 5x + 7a$, find a so that $f(2) = 10$.

10 Given $f(x) = bx^3 - 5x^2 + 2bx + 10$, find b so that $f(\sqrt{3}) = -20$.

11 There are four possible end behaviours for a polynomial function $P(x)$. These are:

as $x \to \infty$, $P(x) \to \infty$ and as $x \to -\infty$, $P(x) \to \infty$ or symbolically $(\nwarrow, \nearrow)$

as $x \to \infty$, $P(x) \to -\infty$ and as $x \to -\infty$, $P(x) \to \infty$ or symbolically $(\nwarrow, \searrow)$

as $x \to \infty$, $P(x) \to -\infty$ and as $x \to -\infty$, $P(x) \to -\infty$ or symbolically $(\swarrow, \searrow)$

as $x \to \infty$, $P(x) \to \infty$ and as $x \to -\infty$, $P(x) \to -\infty$ or symbolically $(\swarrow, \nearrow)$

a) By sketching a graph on your GDC, state the type of end behaviour for each of the polynomial functions below.

(i) $P(x) = 2x^4 - 6x^3 + x^2 + 4x - 1$

(ii) $P(x) = -2x^4 - 6x^3 + x^2 + 4x - 1$

(iii) $P(x) = -6x^3 + x^2 + 4x - 1$

(iv) $P(x) = 6x^3 + x^2 - 4x - 1$

(v) $P(x) = x^2 - 4x - 1$

(vi) $P(x) = -2x^6 + x^5 + 2x^4 - 3x^3 + 4x^2 - x + 1$

(vii) $P(x) = x^5 + 2x^4 - x^3 + x^2 - x + 1$

(viii) $P(x) = -x^5 + 2x^4 - x^3 + x^2 - x + 1$

b) Use your results from a) to write a general statement about how the leading term of a polynomial function, $a_n x^n$, determines what type of end behaviour the graph of the function will display. Be specific about how the characteristics of the coefficient, a_n, and the power, n, of the leading term affect the function's end behaviour.

3.2 Quadratic functions

A **linear function** is a polynomial function of degree one that can be written in the general form $f(x) = ax + b$ where $a \neq 0$. Linear equations were briefly reviewed in Section 1.6. It is clear that any linear function will have a single solution (root) of $x = -\frac{b}{a}$. In essence, this is a formula that gives the zero of any linear polynomial.

In this section, we will focus on **quadratic functions** – functions consisting of a second-degree polynomial that can be written in the form $f(x) = ax^2 + bx + c$ such that $a \neq 0$. You are probably familiar with the quadratic formula that gives the zeros of any quadratic polynomial. We will also investigate other methods of finding zeros of quadratics and consider important characteristics of the graphs of quadratic functions.

The word *quadratic* comes from the Latin word *quadratus* that means four-sided, to make square, or simply a square. *Numerus quadratus* means a square number. Before modern algebraic notation was developed in the 17th and 18th centuries, the geometric figure of a square was used to indicate a number multiplying itself. Hence, raising a number to the power of two (in modern notation) is commonly referred to as the operation of squaring. *Quadratic* then came to be associated with a polynomial of degree two rather than being associated with the number four, as the prefix quad often indicates (e.g. quadruple).

Definition of a quadratic function

If a, b and c are real numbers, and $a \neq 0$, the function $f(x) = ax^2 + bx + c$ is a **quadratic function**. The graph of f is the graph of the equation $y = ax^2 + bx + c$ and is called a **parabola**.

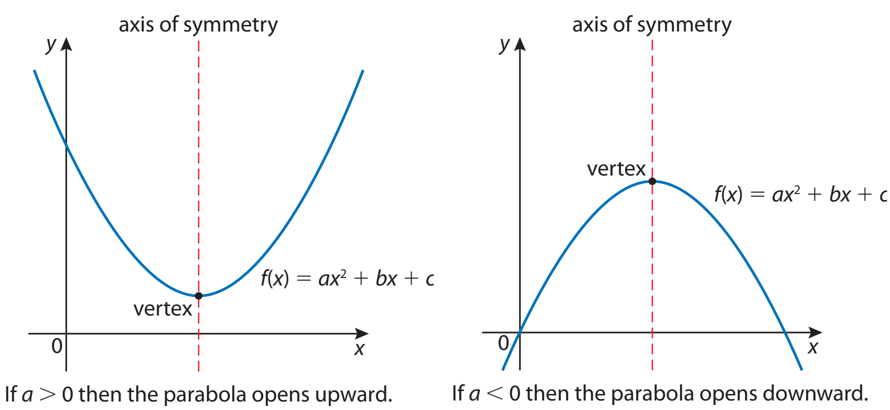


Figure 3.4 'Concave up' and 'concave down' parabolas.

Each parabola is symmetric about a vertical line called its **axis of symmetry.** The axis of symmetry passes through a point on the parabola called the **vertex** of the parabola, as shown in Figure 3.4. If the leading coefficient, a, of the quadratic function $f(x) = ax^2 + bx + c$ is positive, the parabola opens upward (concave up) – and the y-coordinate of the vertex will be a **minimum value** for the function. If the leading coefficient, a, of $f(x) = ax^2 + bx + c$ is negative, the parabola opens downward (concave down) – and the y-coordinate of the vertex will be a **maximum value** for the function.

The graph of $f(x) = a(x - h)^2 + k$

From the previous chapter, we know that the graph of the equation $y = (x + 3)^2 + 2$ can be obtained by translating $y = x^2$ three units to the left and two units up. Being familiar with the shape and position of the graph of $y = x^2$, and knowing the two translations that transform $y = x^2$ to

$y = (x + 3)^2 + 2$, we can easily visualize and/or sketch the graph of $y = (x + 3)^2 + 2$ (see Figure 3.5). We can also determine the axis of symmetry and the vertex of the graph. Figure 3.6 shows that the graph of $y = (x + 3)^2 + 2$ has an axis of symmetry of $x = -3$ and a vertex at $(-3, 2)$. The equation $y = (x + 3)^2 + 2$ can also be written as $y = x^2 + 6x + 11$. Because we can easily identify the vertex of the parabola when the equation is written as $y = (x + 3)^2 + 2$, we often refer to this as the **vertex form** of the quadratic equation, and $y = x^2 + 6x + 11$ as the **general form**.

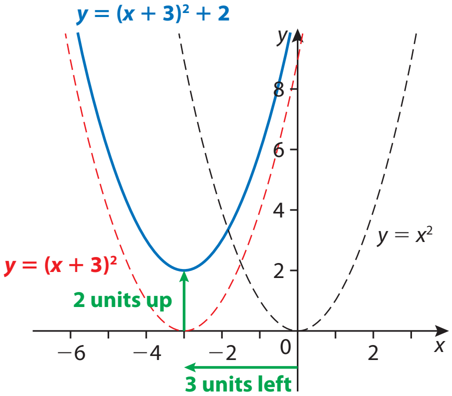


Figure 3.5 Translating $y = x^2$ to give $y = (x + 3)^2 + 2$.

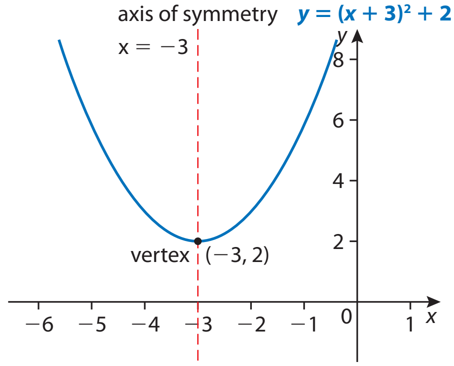


Figure 3.6 The axis of symmetry and the vertex.

- **Hint:** $f(x) = a(x - h)^2 + k$ is sometimes referred to as the **standard form** of a quadratic function.

Vertex form of a quadratic function

If a quadratic function is written in the form $f(x) = a(x - h)^2 + k$, with $a \neq 0$, the graph of f has an axis of symmetry of $x = h$ and a vertex at (h, k).

Completing the square

For visualizing and sketching purposes, it is helpful to have a quadratic function written in vertex form. How do we rewrite a quadratic function written in the form $f(x) = ax^2 + bx + c$ (general form) into the form $f(x) = a(x - h)^2 + k$ (vertex form)? We use the technique of **completing the square.**

For any real number p, the quadratic expression $x^2 + px + \left(\frac{p}{2}\right)^2$ is the square of $\left(x + \frac{p}{2}\right)$. Convince yourself of this by expanding $\left(x + \frac{p}{2}\right)^2$. The technique of *completing the square* is essentially the process of adding a constant to a quadratic expression to make it the square of a binomial. If the coefficient of the quadratic term (x^2) is positive one, the coefficient of the linear term is p, and the constant term is $\left(\frac{p}{2}\right)^2$, then $x^2 + px + \left(\frac{p}{2}\right)^2 = \left(x + \frac{p}{2}\right)^2$ and the square is completed.

Remember that the coefficient of the quadratic term (leading coefficient) must be equal to positive one before completing the square.

Example 3

Find the equation of the axis of symmetry and the coordinates of the vertex of the graph of $f(x) = x^2 - 8x + 18$ by rewriting the function in the form $x^2 + px + \left(\frac{p}{2}\right)^2$.

Solution

To complete the square and get the quadratic expression $x^2 - 8x + 18$ in the form $x^2 + px + \left(\frac{p}{2}\right)^2$, the constant term needs to be $\left(\frac{-8}{2}\right)^2 = 16$.

We need to add 16, but also subtract 16, so that we are adding zero overall and, hence, not changing the original expression.

$f(x) = x^2 - 8x + 16 - 16 + 18$ — Actually adding zero $(-16 + 16)$ to the right side.

$f(x) = x^2 - 8x + 16 + 2$ — $x^2 - 8x + 16$ fits the pattern $x^2 + px + \left(\frac{p}{2}\right)^2$ with $p = -8$.

$f(x) = (x - 4)^2 + 2$ — $x^2 - 8x + 16 = (x - 4)^2$

The axis of symmetry of the graph of f is the vertical line $x = 4$ and the vertex is at $(4, 2)$. See Figure 3.7.

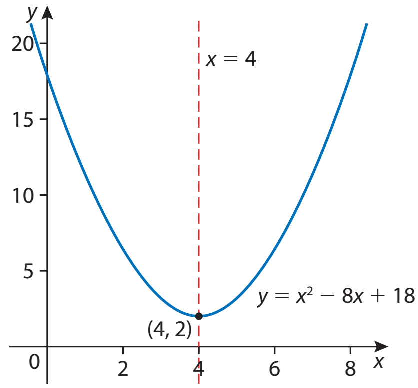


Figure 3.7

Example 4 – Properties of a parabola

For the function $g: x \mapsto -2x^2 - 12x + 7$,

a) find the axis of symmetry and the vertex of the graph

b) indicate the transformations that can be applied to $y = x^2$ to obtain the graph

c) find the minimum or maximum value.

Solution

a) $g: x \mapsto -2\left(x^2 + 6x - \frac{7}{2}\right)$ — Factorize so that the coefficient of the quadratic term is $+1$.

$g: x \mapsto -2\left(x^2 + 6x + 9 - 9 - \frac{7}{2}\right)$ — $p = 6 \Rightarrow \left(\frac{p}{2}\right)^2 = 9$; hence, add $+9 - 9$ (zero)

$g: x \mapsto -2\left[(x + 3)^2 - \frac{18}{2} - \frac{7}{2}\right]$ — $x^2 + 6x + 9 = (x + 3)^2$

$g: x \mapsto -2\left[(x + 3)^2 - \frac{25}{2}\right]$

$g: x \mapsto -2(x + 3)^2 + 25$ — Multiply through by -2 to remove outer brackets.

$g: x \mapsto -2(x - (-3))^2 + 25$ — Express in vertex form: $g: x \mapsto a(x - h)^2 + k$

The axis of symmetry of the graph of g is the vertical line $x = -3$ and the vertex is at $(-3, 25)$. See Figure 3.8.

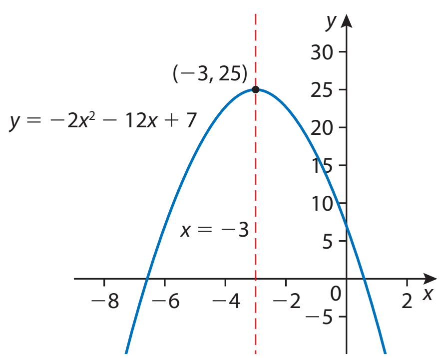


Figure 3.8

b) Since $g: x \mapsto -2x^2 - 12x + 7 = -2(x + 3)^2 + 25$, the graph of g can be obtained by applying the following transformations (in the order given) on the graph of $y = x^2$: horizontal translation of 3 units left;

reflection in the x-axis (parabola opening down); vertical stretch of factor 2; and a vertical translation of 25 units up.

c) The parabola opens down because the leading coefficient is negative. Therefore, g has a maximum and no minimum value. The maximum value is 25 (y-coordinate of vertex) at $x = -3$.

The technique of completing the square can be used to derive the quadratic formula. The following example derives a general expression for the axis of symmetry and vertex of a quadratic function in the general form $f(x) = ax^2 + bx + c$ by completing the square.

Example 5 – Graphical properties of general quadratic functions

Find the axis of symmetry and the vertex for the general quadratic function $f(x) = ax^2 + bx + c$.

Solution

$f(x) = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$ — Factorize so that the coefficient of the x^2 term is $+1$.

$f(x) = a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right]$ — $p = \frac{b}{a} \Rightarrow \left(\frac{p}{2}\right)^2 = \left(\frac{b}{2a}\right)^2$

$f(x) = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right]$ — $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = x + \left(\frac{b}{2a}\right)^2$

$f(x) = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$ — Multiply through by a.

$f(x) = a\left(x - \left(-\frac{b}{2a}\right)\right)^2 + c - \frac{b^2}{4a}$ — Express in vertex form: $f(x) = a(x - h)^2 + k$

This result leads to the following generalization.

Symmetry and vertex of $f(x) = ax^2 + bx + c$

For the graph of the quadratic function $f(x) = ax^2 + bx + c$, the axis of symmetry is the vertical line with the equation $x = -\frac{b}{2a}$ and the vertex has coordinates $\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$.

Check the results for Example 4 using the formulae for the axis of symmetry and vertex. For the function $g: x \mapsto -2x^2 - 12x + 7$:

$$x = -\frac{b}{2a} = -\frac{-12}{2(-2)} = -3 \Rightarrow \text{ axis of symmetry is the vertical line } x = -3$$

$$c - \frac{b^2}{4a} = 7 - \frac{(-12)^2}{4(-2)} = \frac{56}{8} + \frac{144}{8} = 25 \Rightarrow \text{ vertex has coordinates } (-3, 25)$$

These results agree with the results from Example 4.

Zeros of a quadratic function

A specific value for x is a **zero** of a quadratic function $f(x) = ax^2 + bx + c$ if it is a solution (or **root**) to the equation $ax^2 + bx + c = 0$.

As we will observe, every quadratic function will have two zeros although it is possible for the same zero to occur twice (double zero, or double root). The x-coordinate of any point(s) where f crosses the x-axis (y-coordinate is zero) is a **real zero** of the function. A quadratic function can have one, two or no real zeros as Figure 3.9 illustrates. To find non-real zeros we need to extend our search to the set of complex numbers and we will see that a quadratic function with no real zeros will have two distinct **imaginary zeros**. Finding all zeros of a quadratic function requires you to solve quadratic equations of the form $ax^2 + bx + c = 0$. Although $a \neq 0$, it is possible for b or c to be equal to zero. There are five general methods for solving quadratic equations as outlined in Table 3.2 below.

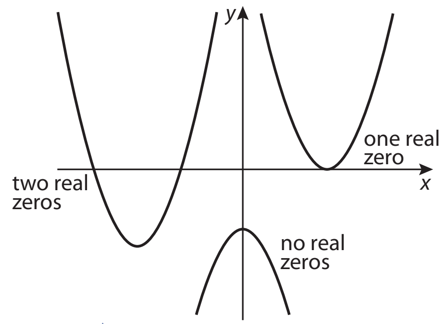
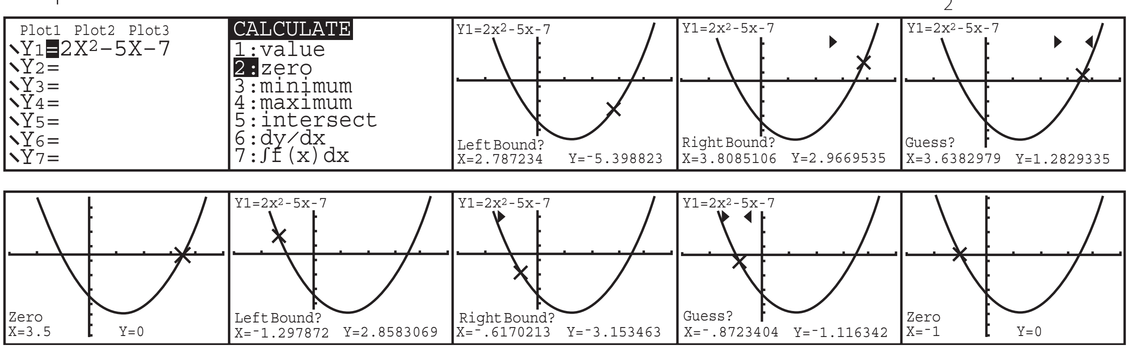


Figure 3.9

Table 3.2 Methods for solving quadratic equations.

Square root Examples	If $a^2 = c$ and $c > 0$, then $a = \pm\sqrt{c}$. $x^2 - 25 = 0$ $(x + 2)^2 = 15$ $x^2 = 25$ $x + 2 = \pm\sqrt{15}$ $x = \pm 5$ $x = -2 \pm\sqrt{15}$
Factorizing Examples	If $ab = 0$, then $a = 0$ or $b = 0$. $x^2 + 3x - 10 = 0$ $x^2 - 7x = 0$ $(x + 5)(x - 2) = 0$ $x(x - 7) = 0$ $x = -5$ or $x = 2$ $x = 0$ or $x = 7$
Completing the square Example	If $x^2 + px + q = 0$, then $x^2 + px + \left(\frac{p}{2}\right)^2 = -q + \left(\frac{p}{2}\right)^2$ which leads to $\left(x + \frac{p}{2}\right)^2 = -q + \frac{p^2}{4}$ and then the square root of both sides (as above). $x^2 - 8x + 5 = 0$ $x^2 - 8x + 16 = -5 + 16$ $(x - 4)^2 = 11$ $x - 4 = \pm\sqrt{11}$ $x = 4 \pm \sqrt{11}$
Quadratic formula Example	If $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. $2x^2 - 3x - 4 = 0$ $x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-4)}}{2(2)}$ $x = \frac{3 \pm \sqrt{41}}{4}$
Graphing Example	Graph the equation $y = ax^2 + bx + c$ on your GDC. Use the calculating features of your GDC to determine the x-coordinates of the point(s) where the parabola intersects the x-axis. *Note*: This method works for finding real solutions, but **not** imaginary solutions. $2x^2 - 5x - 7 = 0$ GDC calculations reveal that the zeros are at $x = \frac{7}{2}$ and $x = -1$



Sum and product of the roots of a quadratic equation

In the next section, the Factor Theorem formally states the relationship between linear factors of the form $x - \alpha$ and the zeros for *any* polynomial.

Consider the quadratic equation $x^2 + 5x - 24 = 0$. This equation can be solved using factorization as follows.

$$x^2 + 5x - 24 = (x + 8)(x - 3) = 0 \Rightarrow x = -8 \text{ or } x = 3$$

Clearly, if $x - \alpha$ is a factor of the quadratic polynomial $ax^2 + bx + c$, then $x = \alpha$ is a root (solution) of the quadratic equation $ax^2 + bx + c = 0$.

Now let us consider the general quadratic equation $ax^2 + bx + c = 0$, whose roots are $x = \alpha$ and $x = \beta$. Given our observation from the previous paragraph, we can write the quadratic equation with roots α and β as:

$$\begin{aligned} ax^2 + bx + c = (x - \alpha)(x - \beta) &= 0 \\ x^2 - \alpha x - \beta x + \alpha\beta &= 0 \\ x^2 - (\alpha + \beta)x + \alpha\beta &= 0 \end{aligned}$$

Since the equation $ax^2 + bx + c = 0$ can also be written as $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$, then:

$$x^2 - (\alpha + \beta)x + \alpha\beta = x^2 + \frac{b}{a}x + \frac{c}{a}$$

Equating coefficients of both sides, gives the following results.

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

Sum and product of the roots of a quadratic equation

For any quadratic equation in the form $ax^2 + bx + c = 0$, the **sum of the roots** of the equation is $-\frac{b}{a}$ and the **product of the roots** is $\frac{c}{a}$. (In the next section, this result is extended to polynomial equations of any degree.)

Example 6

If α and β are the roots of each equation, find the sum, $\alpha + \beta$, and product, $\alpha\beta$, of the roots.

a) $x^2 - 5x + 3 = 0$ b) $3x^2 + 4x - 7 = 0$

Solution

a) For the equation $x^2 - 5x + 3 = 0$, $a = 1$, $b = -5$ and $c = 3$.
Therefore, $\alpha + \beta = -\frac{b}{a} = -\frac{-5}{1} = 5$ and $\alpha\beta = \frac{c}{a} = \frac{3}{1} = 3$.

b) For the equation $3x^2 + 4x - 7 = 0$, $a = 3$, $b = 4$ and $c = -7$.
Therefore, $\alpha + \beta = -\frac{b}{a} = -\frac{4}{3}$ and $\alpha\beta = \frac{c}{a} = \frac{-7}{3}$.

If the sum and product of the roots of a quadratic equation are known, then the equation can be written in the following form: $x^2 -$ (sum of roots)$x +$ (product of roots) $= 0$

Example 7

If α and β are the roots of the equation $2x^2 + 6x - 5 = 0$, find a quadratic equation whose roots are:

a) $2\alpha, 2\beta$ b) $\frac{1}{\alpha + 1}, \frac{1}{\beta + 1}$

Solution

For the equation $2x^2 + 6x - 5 = 0$, $a = 2$, $b = 6$ and $c = -5$.

Thus, $\alpha + \beta = -\frac{b}{a} = -\frac{6}{2} = -3$ and $\alpha\beta = \frac{c}{a} = \frac{-5}{2}$.

a) Sum of the new roots $= 2\alpha + 2\beta = 2(\alpha + \beta) = 2(-3) = -6$.

Thus for the new equation, $-\frac{b}{a} = -6$.

Product of the new roots $= 2\alpha \cdot 2\beta = 4\alpha\beta = 4\left(-\frac{5}{2}\right) = -10$.

Thus for the new equation, $\frac{c}{a} = -10$.

The new equation we are looking for can be written as $ax^2 + bx + c = 0$ or $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$.

Therefore, the quadratic equation with roots $2\alpha, 2\beta$ is $x^2 - (-6)x - 10 = 0$
$\Rightarrow x^2 + 6x - 10 = 0$

b) Sum of the new roots $\frac{1}{\alpha + 1} + \frac{1}{\beta + 1} = \frac{\beta + 1 + \alpha + 1}{(\alpha + 1)(\beta + 1)}$

$$= \frac{\alpha + \beta + 2}{\alpha\beta + \alpha + \beta + 1} = \frac{-3 + 2}{-\frac{5}{2} - 3 + 1} = \frac{-1}{-\frac{9}{2}} = \frac{2}{9}.$$

Thus for the new equation, $-\frac{b}{a} = \frac{2}{9}$.

Product of the new roots $\left(\frac{1}{\alpha + 1}\right)\left(\frac{1}{\beta + 1}\right) = \frac{1}{\alpha\beta + \alpha + \beta + 1}$

$$= \frac{1}{-\frac{5}{2} - 3 + 1} = \frac{1}{-\frac{9}{2}} = -\frac{2}{9}.$$

Thus for the new equation, $\frac{c}{a} = -\frac{2}{9}$.

The new equation we are looking for can be written as $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$.

Therefore, the quadratic equation with roots $\frac{1}{\alpha + 1}, \frac{1}{\beta + 1}$ is $x^2 - \frac{2}{9}x - \frac{2}{9} = 0$ or $9x^2 - 2x - 2 = 0$.

Example 8

Given that the roots of the equation $x^2 - 4x + 2 = 0$ are α and β, find the values of the following expressions.

a) $\alpha^2 + \beta^2$ b) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$

Solution

With $x^2 - 4x + 2 = 0$, $\alpha + \beta = -\frac{b}{a} = -\frac{-4}{1} = 4$ and $\alpha\beta = \frac{c}{a} = \frac{2}{1} = 2$.

Both of the expressions $\alpha^2 + \beta^2$ and $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$ need to be expressed in terms of $\alpha + \beta$ and $\alpha\beta$.

a) $\alpha^2 + \beta^2 = \alpha^2 + 2\alpha\beta + \beta^2 - 2\alpha\beta = (\alpha + \beta)^2 - 2\alpha\beta$

Substituting the values for $\alpha + \beta$ and $\alpha\beta$ from above, gives
$\alpha^2 + \beta^2 = 4^2 - 2 \cdot 2 = 16 - 4 = 12$.

b) $\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\beta^2}{\alpha^2\beta^2} + \frac{\alpha^2}{\alpha^2\beta^2} = \frac{\alpha^2 + \beta^2}{(\alpha\beta)^2}$

From part a) we know that $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$. Substituting this into the numerator gives:

$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$ Then substituting the values for $\alpha + \beta$ and $\alpha\beta$ from above, gives:

$$= \frac{4^2 - 2 \cdot 2}{2^2} = \frac{12}{4} = 3$$

Therefore, $\frac{1}{\alpha^2} + \frac{1}{\beta^2} = 3$.

The quadratic formula and the discriminant

The expression that is beneath the radical sign in the quadratic formula, $b^2 - 4ac$, determines whether the zeros of a quadratic function are real or imaginary. Because it acts to 'discriminate' between the types of zeros, $b^2 - 4ac$ is called the **discriminant**. It is often labelled with the Greek letter Δ (delta). The value of the discriminant can also indicate if the zeros are equal and if they are rational.

The discriminant and the nature of the zeros of a quadratic function

For the quadratic function $f(x) = ax^2 + bx + c$, $(a \neq 0)$ where a, b and c are real numbers:

If $\Delta = b^2 - 4ac > 0$, then f has two distinct real zeros, and the graph of f intersects the x-axis twice.

If $\Delta = b^2 - 4ac = 0$, then f has one real zero (double root), and the graph of f intersects the x-axis once (i.e. it is tangent to the x-axis).

If $\Delta = b^2 - 4ac < 0$, then f has two conjugate imaginary zeros, and the graph of f does not intersect the x-axis.

In the special case when a, b and c are integers and the discriminant is the square of an integer (a *perfect square*), the polynomial $ax^2 + bx + c$ has two distinct **rational zeros**.

When the discriminant is zero then the solution of a quadratic function is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{0}}{2a} = -\frac{b}{2a}$. As mentioned, this solution of $-\frac{b}{2a}$ is called a double zero (or root) which can also be described as a **zero of multiplicity of 2**. If a and b are integers then the zero $-\frac{b}{2a}$ will be rational.

When we solve polynomial functions of higher degree later this chapter, we will encounter zeros of higher multiplicity.

• **Hint:** Remember that the **roots** of a polynomial equation are those values of x for which $P(x) = 0$. These values of x are called the **zeros** of the polynomial P.

Factorable quadratics

If the zeros of a quadratic polynomial are rational – either two distinct zeros or two equal zeros (double zero/root) – then the polynomial is factorable. That is, if $ax^2 + bx + c$ has rational zeros then $ax^2 + bx + c = (mx + n)(px + q)$ where m, n, p and q are rational numbers.

Example 9 – Using discriminant to determine the nature of the roots of a quadratic equation

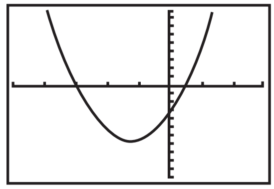
Use the discriminant to determine how many real roots each equation has. Visually confirm the result by graphing the corresponding quadratic function for each equation on your GDC.

a) $2x^2 + 5x - 3 = 0$ b) $4x^2 - 12x + 9 = 0$ c) $2x^2 - 5x + 6 = 0$

Solution

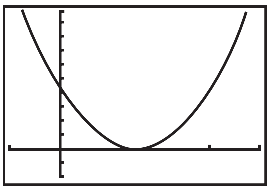
a) The discriminant is $\Delta = 5^2 - 4(2)(-3) = 49 > 0$. Therefore, the equation has two distinct real roots. This result is confirmed by the graph of the quadratic function $y = 2x^2 + 5x - 3$ that clearly shows it intersecting the x-axis twice. Also since $\Delta = 49$ is a perfect square then the two roots are also rational and the quadratic polynomial $2x^2 + 5x - 3 = 0$ is factorable: $2x^2 + 5x - 3 = (2x - 1)(x + 3) = 0$. Thus, the two rational roots are $x = \frac{1}{2}$ and $x = -3$.

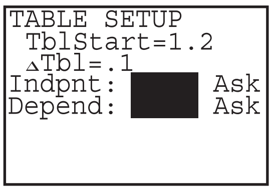
$y = 2x^2 + 5x - 3$

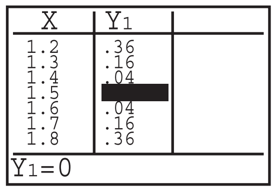


b) The discriminant is $\Delta = (-12)^2 - 4(4)(9) = 0$. Therefore, the equation has one rational root (a double root). The graph on the GDC of $y = 4x^2 - 12x + 9$ appears to intersect the x-axis at only one point. We can be more confident with this conclusion by investigating further – for example, tracing or looking at a table of values on the GDC.

$y = 4x^2 - 12x + 9$







Also, since the root is rational ($\Delta = 0$), the polynomial $4x^2 - 12x + 9$ must be factorable.

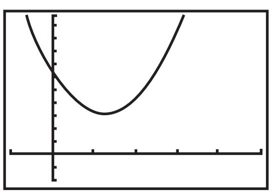
$4x^2 - 12x + 9 = (2x - 3)(2x - 3) = \left[2\left(x - \frac{3}{2}\right)2\left(x - \frac{3}{2}\right)\right] = 4\left(x - \frac{3}{2}\right)^2 = 0$

There are two equal linear factors which means there are two equal rational zeros – both equal to $\frac{3}{2}$ in this case.

c) The discriminant is $\Delta = (-5)^2 - 4(2)(6) = -23 < 0$. Therefore, the equation has no real roots. This result is confirmed by the graph of the quadratic function $y = 2x^2 - 5x + 6$ that clearly shows that the graph does not intersect the x-axis. The equation will have two imaginary roots.

● **Hint:** If a quadratic polynomial has a zero of multiplicity 2($\Delta = 0$), as in Example 6 b), then not only is the polynomial factorable but its factorization will contain two equal linear factors. In such a case then $ax^2 + bx + c = a(x - p)^2$ where $x - p$ is the linear factor and $x = p$ is the rational zero.

$y = 2x^2 - 5x + 6$



Example 10 – The discriminant and number of real zeros

For $4x^2 + 4kx + 9 = 0$, determine the value(s) of k so that the equation has: a) one real zero, b) two distinct real zeros, and c) no real zeros.

Solution

a) For one real zero $\Delta = (4k)^2 - 4(4)(9) = 0 \Rightarrow 16k^2 - 144 = 0$

$\Rightarrow 16k^2 = 144 \Rightarrow k^2 = 9 \Rightarrow k = \pm 3$

b) For two distinct real zeros $\Delta = (4k)^2 - 4(4)(9) > 0 \Rightarrow 16k^2 > 144$
$\Rightarrow k^2 > 9 \Rightarrow k < -3$ or $k > 3$

c) For no real zeros $\Delta = (4k)^2 - 4(4)(9) < 0 \Rightarrow 16k^2 < 144 \Rightarrow k^2 < 9$
$\Rightarrow k > -3$ and $k < 3 \Rightarrow -3 < k < 3$

Example 11 – Conjugate imaginary solutions

Find the zeros of the function $g: x \to 2x^2 - 4x + 7$.

Solution

Solve the equation $2x^2 - 4x + 7 = 0$ using the quadratic formula with $a = 2, b = -4, c = 7$.

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(7)}}{2(2)} = \frac{4 \pm \sqrt{-40}}{4} = \frac{4 \pm \sqrt{4}\sqrt{-1}\sqrt{10}}{4}$$

$$= \frac{4 \pm 2i\sqrt{10}}{4} = 1 \pm \frac{i\sqrt{10}}{2}$$

The two zeros of g are $1 + \frac{\sqrt{10}}{2}i$ and $1 - \frac{\sqrt{10}}{2}i$.

Note that the imaginary zeros are written in the form $a + bi$ (introduced in Section 1.1) and that they clearly are a pair of conjugates, i.e. fitting the pattern $a + bi$ and $a - bi$.

Number of complex zeros of a quadratic polynomial
Every quadratic polynomial has exactly two complex zeros, provided that a zero of multiplicity 2 (two equal zeros) is counted as two zeros.

• **Hint:** Recall from Section 1.1 that the real numbers and the imaginary numbers are distinct subsets of the complex numbers. A complex number can be either real $\left(\text{e.g. } -7, \frac{\pi}{2}, 3 - \sqrt{2}\right)$ or imaginary (e.g. $4i$, $2 + i\sqrt{5}$).

The graph of $f(x) = a(x - p)(x - q)$

If a quadratic function is written in the form $f(x) = a(x - p)(x - q)$ then we can easily identify the x-intercepts of the graph of f. Consider that $f(p) = a(p - p)(p - q) = a(0)(p - q) = 0$ and that $f(q) = a(q - p)(q - q) = a(q - p)(0) = 0$. Therefore, the quadratic function $f(x) = a(x - p)(x - q)$ will intersect the x-axis at the points $(p, 0)$ and $(q, 0)$. We need to factorize in order to rewrite a quadratic function in the form $f(x) = ax^2 + bx + c$ to the form $f(x) = a(x - p)(x - q)$. Hence, $f(x) = a(x - p)(x - q)$ can be referred to as the **factorized** form of a quadratic function. Recalling the symmetric nature of a parabola, it is clear that the x-intercepts $(p, 0)$ and $(q, 0)$ will be equidistant from the axis of symmetry (see Figure 3.10). As a result, the equation of the axis of symmetry and the x-coordinate of the vertex of the parabola can be found from finding the average of p and q.

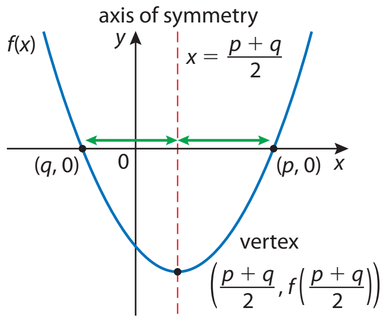
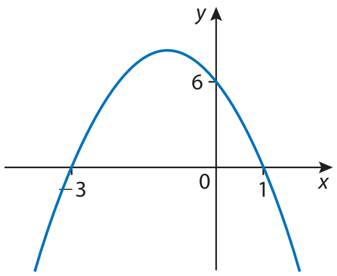


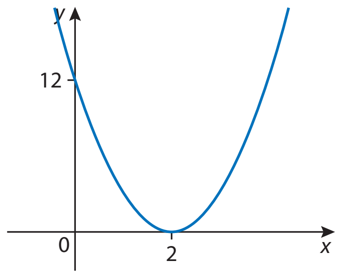
Figure 3.10

Factorized form of a quadratic function
If a quadratic function is written in the form $f(x) = a(x - p)(x - q)$, with $a \neq 0$, the graph of f has x-intercepts at $(p, 0)$ and $(q, 0)$, an axis of symmetry with equation $x = \frac{p+q}{2}$, and a vertex at $\left(\frac{p+q}{2}, f\left(\frac{p+q}{2}\right)\right)$.

Example 12

Find the equation of each quadratic function from the graph in the form $f(x) = a(x - p)(x - q)$ and also in the form $f(x) = ax^2 + bx + c$.

a) 

b) 

Solution

a) Since the x-intercepts are -3 and 1 then $y = a(x + 3)(x - 1)$. The y-intercept is 6, so when $x = 0$, $y = 6$. Hence, $6 = a(0 + 3)(0 - 1) = -3a \Rightarrow a = -2$ ($a < 0$ agrees with the fact that the parabola is opening down). The function is $f(x) = -2(x + 3)(x - 1)$, and expanding to remove brackets reveals that the function can also be written as $f(x) = -2x^2 - 4x + 6$.

b) The function has one x-intercept at 2 (double root), so $p = q = 2$ and $y = a(x - 2)(x - 2) = a(x - 2)^2$. The y-intercept is 12, so when $x = 0$, $y = 12$. Hence, $12 = a(0 - 2)^2 = 4a \Rightarrow a = 3$ ($a > 0$ agrees with the parabola opening up). The function is $f(x) = 3(x - 2)^2$. Expanding reveals that the function can also be written as $f(x) = 3x^2 - 12x + 12$.

Example 13

The graph of a quadratic function intersects the x-axis at the points $(-6, 0)$ and $(-2, 0)$ and also passes through the point $(2, 16)$. a) Write the function in the form $f(x) = a(x - p)(x - q)$. b) Find the vertex of the parabola. c) Write the function in the form $f(x) = a(x - h)^2 + k$.

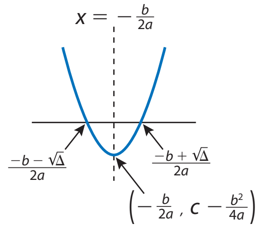
Solution

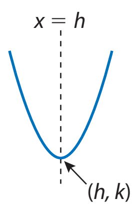
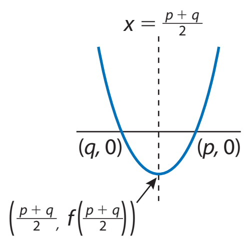
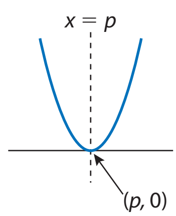
a) The x-intercepts of -6 and -2 gives $f(x) = a(x + 6)(x + 2)$. Since f passes through $(2, 16)$, then $f(2) = 16 \Rightarrow f(2) = a(2 + 6)(2 + 2) = 16$ $\Rightarrow 32a = 16 \Rightarrow a = \frac{1}{2}$. Therefore, $f(x) = \frac{1}{2}(x + 6)(x + 2)$.

b) The x-coordinate of the vertex is the average of the x-intercepts. $x = \dfrac{-6 - 2}{2} = -4$, so the y-coordinate of the vertex is $y = f(-4) = \frac{1}{2}(-4 + 6)(-4 + 2) = -2$. Hence, the vertex is $(-4, -2)$.

c) In vertex form, the quadratic function is $f(x) = \frac{1}{2}(x + 4)^2 - 2$.

Table 3.3 Review of properties of quadratics.

<table>
<tr><th>Quadratic function, $a \neq 0$</th><th>Graph of function</th><th>Results</th></tr>
<tr><td>General form
$f(x) = ax^2 + bx + c$
$\Delta = b^2 - 4ac$ (discriminant)</td><td>Parabola opens up if $a > 0$
Parabola opens down if $a < 0$


If $\Delta \geqslant 0$, f has x-intercept(s)
If $\Delta < 0$, f has no x-intercept(s)</td><td>Axis of symmetry is $x = -\frac{b}{2a}$
If $\Delta \geqslant 0$, f has x-intercept(s):
$\left(\frac{-b \pm \sqrt{\Delta}}{2a}, 0\right)$
Vertex is: $\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$</td></tr>
<tr><td>Vertex form
$f(x) = a(x-h)^2 + k$</td><td>
</td><td>Axis of symmetry is $x = h$
Vertex is (h, k)</td></tr>
<tr><td>Factorized form
(two distinct rational zeros)
$f(x) = a(x-p)(x-q)$</td><td>
</td><td>Axis of symmetry is $x = \frac{p+q}{2}$
x-intercepts are: $(p, 0)$ and $(q, 0)$</td></tr>
<tr><td>Factorized form
(one rational zero)
$f(x) = a(x-p)^2$</td><td>
</td><td>Axis of symmetry is $x = p$
Vertex and x-intercept is $(p, 0)$</td></tr>
</table>

Exercise 3.2

For each of the quadratic functions f in questions 1–5, find the following:

a) the axis of symmetry and the vertex, by algebraic methods

b) the transformation(s) that can be applied to $y = x^2$ to obtain the graph of $y = f(x)$

c) the minimum or maximum value of f.

Check your results using your GDC.

1 $f: x \mapsto x^2 - 10x + 32$

2 $f: x \mapsto x^2 + 6x + 8$

3 $f: x \mapsto -2x^2 - 4x + 10$

4 $f: x \mapsto 4x^2 - 4x + 9$

5 $f: x \mapsto \frac{1}{2}x^2 + 7x + 26$

In questions 6–13, solve the quadratic equation using factorization.

6 $x^2 + 2x - 8 = 0$

7 $x^2 = 3x + 10$

8 $6x^2 - 9x = 0$

9 $6 + 5x = x^2$

10 $x^2 + 9 = 6x$

11 $3x^2 + 11x - 4 = 0$

12 $3x^2 + 18 = 15x$

13 $9x - 2 = 4x^2$

In questions 14–19, use the method of completing the square to solve the quadratic equation.

14 $x^2 + 4x - 3 = 0$

15 $x^2 - 4x - 5 = 0$

16 $x^2 - 2x + 3 = 0$

17 $2x^2 + 16x + 6 = 0$

18 $x^2 + 2x - 8 = 0$

19 $-2x^2 + 4x + 9 = 0$

20 Let $f(x) = x^2 - 4x - 1$. a) Use the quadratic formula to find the zeros of the function. b) Use the zeros to find the equation for the axis of symmetry of the parabola. c) Find the minimum or maximum value of f.

In questions 21–24, determine the number of real solutions to each equation.

21 $x^2 + 3x + 2 = 0$

22 $2x^2 - 3x + 2 = 0$

23 $x^2 - 1 = 0$

24 $2x^2 - \frac{9}{4}x + 1 = 0$

25 Find the value(s) of p for which the equation $2x^2 + px + 1 = 0$ has one real solution.

26 Find the value(s) of k for which the equation $x^2 + 4x + k = 0$ has two distinct real solutions.

27 The equation $x^2 - 4kx + 4 = 0$ has two distinct real solutions. Find the set of all possible values of k.

28 Find all possible values of m so that the graph of the function $g: x \mapsto mx^2 + 6x + m$ does not touch the x-axis.

29 Find the range of values of k such that $3x^2 - 12x + k > 0$ for all real values of x. (Hint: Consider what must be true about the zeros of the quadratic equation $y = 3x^2 - 12x + k$.)

30 Prove that the expression $x - 2 - x^2$ is negative for all real values of x.

In questions 31 and 32, find a quadratic function in the form $y = ax^2 + bx + c$ that satisfies the given conditions.

31 The function has zeros of $x = -1$ and $x = 4$ and its graph intersects the y-axis at $(0, 8)$.

32 The function has zeros of $x = \frac{1}{2}$ and $x = 3$ and its graph passes through the point $(-1, 4)$.

33 Find the range of values for k in order for the equation $2x^2 + (3 - k)x + k + 3 = 0$ to have two imaginary solutions.

34 For what values of m does the function $f(x) = 5x^2 - mx + 2$ have two distinct real zeros?

35 The graph of a quadratic function passes through the points $(-3, 10)$, $(\frac{1}{4}, -\frac{9}{16})$ and $(1, 6)$. Express the function in the form $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$.

36 The maximum value of the function $f(x) = ax^2 + bx + c$ is 10.
Given that $f(3) = f(-1) = 2$, find $f(2)$.

37 Find the values of x for which $4x + 1 < x^2 + 4$.

38 Show that there is no real value t for which the equation $2x^2 + (2 - t)x + t^2 + 3 = 0$ has real roots.

39 Show that the two roots of $ax^2 - a^2x - x + a = 0$ are reciprocals of each other.

40 Find the sum and product of the roots for each of the following quadratic equations.

a) $2x^2 + 6x - 5 = 0$ b) $x^2 = 1 - 3x$ c) $4x^2 - 6 = 0$

d) $x^2 + ax - 2a = 0$ e) $m(m - 2) = 4(m + 1)$ f) $3x - \frac{2}{x} = 1$

41 The roots of the equation $2x^2 - 3x + 6 = 0$ are α and β. Find a quadratic equation with integral coefficients whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.

42 If α and β are the roots of the equation $3x^2 + 5x + 4 = 0$, find the values of the following expressions.

a) $\alpha^2 + \beta^2$ b) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

c) $\alpha^3 + \beta^3$ [Hint: factorise $\alpha^3 + \beta^3$ into a product of a binomial and a trinomial.]

43 Consider the quadratic equation $x^2 + 8x + k = 0$ where k is a constant.

a) Find both roots of the equation given that one root of the equation is three times the other.

b) Find the value of k.

44 The roots of the equation $x^2 + x + 4 = 0$ are α and β.

a) Without solving the equation, find the value of the expression $\frac{1}{\alpha} + \frac{1}{\beta}$.

b) Find a quadratic equation whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

45 If α and β are roots of the quadratic equation $5x^2 - 3x - 1 = 0$, find a quadratic equation with integral coefficients which have the roots:

a) $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$ b) $\frac{\alpha^2}{\beta}$ and $\frac{\beta^2}{\alpha}$

3.3 Zeros, factors and remainders

Finding the zeros of polynomial functions is a feature of many problems in algebra, calculus and other areas of mathematics. In our analysis of quadratic functions in the previous section, we saw the connection between the graphical and algebraic approaches to finding zeros. Information obtained from the graph of a function can be used to help find its zeros and, conversely, information about the zeros of a polynomial function can be used to help sketch its graph. Results and observations from the last section lead us to make some statements about real zeros of all polynomial functions. Later in this section we will extend our consideration to imaginary zeros. The following box summarizes what we have observed thus far about the zeros of polynomial functions.

Real zeros of polynomial functions

If P is a polynomial function and c is a real number, then the following statements are equivalent.

- $x = c$ is a zero of the function P.
- $x = c$ is a solution (or root) of the polynomial equation $P(x) = 0$.
- $x - c$ is a linear factor of the polynomial P.
- $(c, 0)$ is an x-intercept of the graph of the function P.

Polynomial division

As with integers, finding the factors of polynomials is closely related to dividing polynomials. An integer n is **divisible** by another integer m if m is a factor of n. If n is not divisible by m we can use the process of **long division** to find the quotient of the numbers and the remainder. For example, let's use long division to divide 485 by 34.

$$\begin{array}{r} 14 \\ 34\overline{)485} \\ \underline{34} \\ 145 \\ \underline{136} \\ 9 \end{array} \qquad \text{check:} \quad \begin{array}{rl} 14 & \text{quotient} \\ \underline{\times\,34} & \text{divisor} \\ 56 & \\ \underline{420} & \\ 476 & \\ \underline{+\;\;9} & \text{remainder} \\ 485 & \text{dividend} \end{array}$$

The number 485 is the **dividend**, 34 is the **divisor**, 14 is the **quotient** and 9 is the **remainder**. The long division process (or algorithm) stops when a remainder is less than the divisor. The procedure shown above for checking the division result may be expressed as

$$485 = 34 \times 14 + 9$$

or in words as

$$\text{dividend} = \text{divisor} \times \text{quotient} + \text{remainder}$$

The process of division for polynomials is similar to that for integers. If a polynomial $D(x)$ is a factor of polynomial $P(x)$, then $P(x)$ is divisible by $D(x)$. However, if $D(x)$ is not a factor of $P(x)$ then we can use a **long division algorithm for polynomials** to find a quotient polynomial $Q(x)$ and a remainder polynomial $R(x)$ such that $P(x) = D(x) \cdot Q(x) + R(x)$. In the same way that the remainder must be less than the divisor when dividing integers, the remainder must be a polynomial of a lower degree than the divisor when dividing polynomials. Consequently, when the divisor is a linear polynomial (degree of 1) the remainder must be of degree 0, i.e. a constant.

Hint: A common error when performing long division with polynomials is to add rather than subtract during each cycle of the process.

Example 14

Find the quotient $Q(x)$ and remainder $R(x)$ when $P(x) = 2x^3 - 5x^2 + 6x - 3$ is divided by $D(x) = x - 2$.

Solution

$$
\begin{array}{rll}
 & 2x^2 - x + 4 & \\
x - 2 \,\big) & 2x^3 - 5x^2 + 6x - 3 & \\
 & \underline{2x^3 - 4x^2} & \leftarrow 2x^2(x-2) \\
 & -x^2 + 6x & \leftarrow \text{Subtract} \\
 & \underline{-x^2 + 2x} & \leftarrow -x(x-2) \\
 & 4x - 3 & \leftarrow \text{Subtract} \\
 & \underline{4x - 8} & \leftarrow 4(x-2) \\
 & 5 & \leftarrow \text{Subtract}
\end{array}
$$

Thus, the quotient $Q(x)$ is $2x^2 - x + 4$ and the remainder is 5. Therefore, we can write

$$2x^3 - 5x^2 + 6x - 3 = (x - 2)(2x^2 - x + 4) + 5$$

This equation provides a means to check the result by expanding and simplifying the right side and verifying it is equal to the left side.

$$
\begin{aligned}
2x^3 - 5x^2 + 6x - 3 &= (x - 2)(2x^2 - x + 4) + 5 \\
&= (2x^3 - x^2 + 4x - 4x^2 + 2x - 8) + 5 \\
&= 2x^3 - 5x^2 + 6x - 3
\end{aligned}
$$

Taking the identity $P(x) = D(x) \cdot Q(x) + R(x)$ and dividing both sides by $D(x)$ produces the equivalent identity $\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$.

Hence, the result for Example 14 could also be written as

$$\frac{2x^3 - 5x^2 + 6x - 3}{x - 2} = 2x^2 - x + 4 + \frac{5}{x - 2}.$$

Note that writing the result in this manner is the same as rewriting $17 = 5 \times 3 + 2$ as $\frac{17}{5} = 3 + \frac{2}{5}$, which we commonly write as the 'mixed number' $3\frac{2}{5}$.

Hint: When performing long division with polynomials it is necessary to write all polynomials so that the powers (exponents) of the terms are in descending order. Example 12 illustrates that if there are any 'missing' terms then they have a coefficient of zero and a zero must be included in the appropriate location in the division scheme.

Example 15

Divide $f(x) = 4x^3 - 31x - 15$ by $2x + 5$, and use the result to factor $f(x)$ completely.

Solution

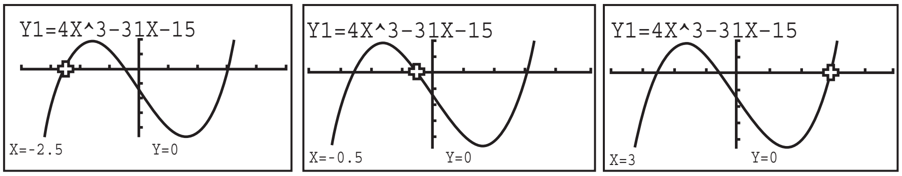
$$
\begin{array}{rl}
 & 2x^2 - 5x - 3 \\
2x + 5 \,\big) & 4x^3 + 0x^2 - 31x - 15 \\
 & \underline{4x^3 + 10x^2} \\
 & -10x^2 - 31x \\
 & \underline{-10x^2 - 25x} \\
 & -6x - 15 \\
 & \underline{-6x - 15} \\
 & 0
\end{array}
$$

Thus $f(x) = 4x^3 - 31x - 15 = (2x + 5)(2x^2 - 5x - 3)$

… and factorizing the quadratic quotient (also a factor of $f(x)$), gives

$$f(x) = 4x^3 - 31x - 15 = (2x + 5)(2x^2 - 5x - 3)$$
$$= (2x + 5)(2x + 1)(x - 3)$$

This factorization would lead us to believe that the three zeros of $f(x)$ are $x = -\frac{5}{2}$, $x = -\frac{1}{2}$ and $x = 3$. Graphing $f(x)$ on our GDC and using the 'trace' feature confirms that all three values are zeros of the cubic polynomial.



Division algorithm for polynomials

If $P(x)$ and $D(x)$ are polynomials such that $D(x) \neq 0$, and the degree of $D(x)$ is less than or equal to the degree of $P(x)$, then there exist unique polynomials $Q(x)$ and $R(x)$ such that

$$P(x) = D(x) \cdot Q(x) + R(x)$$

dividend divisor quotient remainder

and where $R(x)$ is either zero or of degree less than the degree of $D(x)$.

Remainder and factor theorems

As illustrated by Examples 14 and 15, we commonly divide polynomials of higher degree by linear polynomials. By doing so we can often uncover zeros of polynomials as occurred in Example 15. Let's look at what happens to the division algorithm when the divisor $D(x)$ is a linear polynomial of the form $x - c$. Since the degree of the remainder $R(x)$ must be less than the degree of the divisor (degree of one in this case) then the remainder will be a constant, simply written as R. Then the division algorithm for a linear divisor is the identity:

$$P(x) = (x - c) \cdot Q(x) + R$$

If we evaluate the polynomial function P at the number $x = c$, we obtain

$$P(c) = (c - c) \cdot Q(c) + R = 0 \cdot Q(c) + R = R$$

Thus the remainder R is equal to $P(c)$, the value of the polynomial P at $x = c$. Because this is true for any polynomial P and any linear divisor $x - c$, we have the following theorem.

The remainder theorem

If a polynomial function $P(x)$ is divided by $x - c$, then the remainder is the value $P(c)$.

Example 16

What is the remainder when $g(x) = 2x^3 + 5x^2 - 8x + 3$ is divided by $x + 4$?

Solution

The linear polynomial $x + 4$ is equivalent to $x - (-4)$. Applying the remainder theorem, the required remainder is equal to the value of $g(-4)$.

$$g(-4) = 2(-4)^3 + 5(-4)^2 - 8(-4) + 3 = 2(-64) + 5(16) + 32 + 3$$
$$= -128 + 80 + 35 = -13$$

Therefore, when the polynomial function $g(x)$ is divided by $x + 4$ the remainder is -13.

Figure 3.11 Connection between synthetic substitution and long division.

$$\begin{array}{c|cccc}
-4 & 2 & 5 & -8 & 3 \\
 & & -8 & 12 & -16 \\
\hline
 & 2 & -3 & 4 & -13 = g(-4)
\end{array}$$

$$\begin{array}{r}
2x^2 - 3x + 4 \\
x + 4\overline{)2x^3 + 5x^2 - 8x + 3} \\
\underline{2x^3 + 8x^2} \qquad\qquad \\
-3x^2 - 8x \qquad \\
\underline{-3x^2 - 12x} \qquad \\
4x + 3 \\
\underline{4x + 16} \\
-13
\end{array}$$

We found the value of $g(-4)$ in Example 16 by directly substituting -4 into $g(x)$. Alternatively, we could have used the efficient method of synthetic substitution that we developed in Section 3.1 to evaluate $g(-4)$.

We could also have found the remainder by performing long division, which is certainly the least efficient method. However, there is a very interesting and helpful connection between the process of long division with a linear divisor and synthetic substitution.

Not only does synthetic substitution find the value of the remainder, but the numbers in the bottom row preceding the remainder (shown in red in Figure 3.11) are the same as the coefficients of the quotient (also in red) found from the long division process. Clearly, synthetic substitution is the most efficient method for finding the remainder *and* quotient when dividing a polynomial by a linear polynomial in the form $x - c$. When this method is used to find a quotient and remainder we refer to it as **synthetic division**.

The numbers in the last row of the synthetic substitution process give both the remainder and the coefficients of the quotient when a polynomial is divided by a linear polynomial in the form $x - c$.

A consequence of the remainder theorem is the factor theorem, which also follows intuitively from our discussion in the previous section about the zeros and factors of quadratic functions. It formalizes the relationship between zeros and linear factors of all polynomial functions with real coefficients.

It is important to understand that the factor theorem is a **biconditional** statement of the form 'A if and only if B'. Such a statement is true in either 'direction'; that is, 'If A then B', and also 'If B then A' – usually abbreviated $A \rightarrow B$ and $B \rightarrow A$, respectively.

The factor theorem

A polynomial function $P(x)$ has a factor $x - c$ if and only if $P(c) = 0$.

To illustrate the efficiency of synthetic division, let's answer the same problem posed in Example 14 (solution reproduced in Figure 3.12) in Example 17.

Example 17

Find the quotient $Q(x)$ and remainder $R(x)$ when $P(x) = 2x^3 - 5x^2 + 6x - 3$ is divided by $D(x) = x - 2$.

Solution

Using synthetic division

$$
\begin{array}{r|rrrr}
2 & 2 & -5 & 6 & -3 \\
  &   & 4 & -2 & 8 \\
\hline
  & \underbrace{2 \quad -1 \quad 4}_{\text{coefficients of the quotient}} & & & 5 \leftarrow \text{remainder}
\end{array}
$$

$$
\begin{array}{rl}
2x^2 - x + 4 & \\
x - 2\overline{)2x^3 - 5x^2 + 6x - 3} & \\
\underline{2x^3 - 4x^2} & \leftarrow 2x^2(x-2) \\
-x^2 + 6x & \leftarrow \text{Subtract} \\
\underline{-x^2 + 2x} & \leftarrow -x(x-2) \\
4x - 3 & \leftarrow \text{Subtract} \\
\underline{4x - 8} & \leftarrow 4(x-2) \\
5 & \leftarrow \text{Subtract}
\end{array}
$$

The quotient $Q(x)$ is $2x^2 - x + 4$ and the remainder is 5.

Figure 3.12 Solution for Example 14.

Since a divisor of degree 1 is dividing a polynomial of degree 3 then the quotient must be of degree 2 and, with all polynomials written so that their terms are descending in powers (exponents), we know that the numbers in the bottom row of the synthetic division scheme are the coefficients of a quadratic polynomial. Hence, the quotient is $2x^2 - x + 4$ and the remainder is 5.

When one or more zeros of a given polynomial are known, applying the factor theorem and synthetic division is a very effective strategy to aid in finding factors and zeros of the polynomial.

Example 18

Given that $x = -\frac{1}{2}$ and $x = 8$ are zeros of the polynomial function $h(x) = x^4 - \frac{15}{2}x^3 - 30x - 16$, find the other two zeros of $h(x)$.

Solution

From the factor theorem, it follows that $x + \frac{1}{2}$ and $x - 8$ are factors of $h(x)$. Dividing the 4th degree polynomial by the two linear factors in succession will yield a quadratic factor. We can find the zeros of this quadratic factor by using known factorizing techniques or by applying the quadratic formula.

$$
\begin{array}{r|rrrrr}
-\frac{1}{2} & 1 & -\frac{15}{2} & 0 & -30 & -16 \\
 & & -\frac{1}{2} & 4 & -2 & 16 \\
\hline
8 & 1 & -8 & 4 & -32 & 0 \\
 & & 8 & 0 & 32 & \\
\hline
 & 1 & 0 & 4 & 0 &
\end{array}
$$

This row shows that $x^4 - \frac{15}{2}x^3 - 30x - 16 = (x + \frac{1}{2})(x^3 - 8x^2 + 4x - 32)$.

This row shows that $x^3 - 8x^2 + 4x - 32 = (x - 8)(x^2 + 4)$.

● Hint: Example 18 indicates that if we divide the quartic polynomial $x^4 - \frac{15}{2}x^3 - 30x - 16$ by $x^2 + 4$ the remainder will be zero, since $x^2 + 4$ is a factor. Synthetic division *only* works for linear divisors of the form $x - c$ so this division could only be done by using the long division process.

Hence, $x^4 - \frac{15}{2}x^3 - 30x - 16 = (x + \frac{1}{2})(x - 8)(x^2 + 4)$.

The zeros of the quadratic factor $x^2 + 4$ must also be zeros of $h(x)$.

$$x^2 + 4 = 0 \Rightarrow x^2 = -4 \Rightarrow x = \pm\sqrt{-4} \Rightarrow x = \pm\sqrt{4}\,\sqrt{-1} \Rightarrow x = \pm 2i$$

Therefore, the other two remaining zeros of $h(x)$ are $x = 2i$ and $x = -2i$.

Note that the two imaginary zeros, $x = 2i$ and $x = -2i$, of the polynomial in Example 18 are a pair of conjugates. In the previous section we asserted that imaginary zeros of a quadratic polynomial always come in conjugate pairs. Although it is beyond the scope of this book to prove it, we will accept that this is true for imaginary zeros of any polynomial.

Conjugate zeros

If a polynomial P has real coefficients, and if the complex number $z = a + bi$ is a zero of P, then its conjugate $z^* = a - bi$ is also a zero of P.

Example 19

Given that $2 - 3i$ is a zero of the polynomial $5x^3 - 19x^2 + 61x + 13$, find all remaining zeros of the polynomial.

Solution

Firstly, we need to consider what is the maximum number of zeros that the cubic polynomial can have. In the previous section we stated that every quadratic polynomial has exactly two complex zeros. It is reasonable to conjecture that a cubic will have three complex zeros. Since $2 - 3i$ is a zero, then $2 + 3i$ must also be a zero; and the third zero must be a real number. Although not explicitly stated in the remainder and factor theorems, both theorems are true for linear polynomials $x - c$ where the number c is real *or* imaginary, i.e. it can be any complex number. Therefore, the cubic polynomial has factors $x - (2 - 3i)$ and $x - (2 + 3i)$. Rather than attempting to divide the cubic polynomial by one of these factors, let's find the product of these factors and use it as a divisor.

$$\begin{aligned}
[x - (2 - 3i)][x - (2 + 3i)] &= [x - 2 + 3i][x - 2 - 3i] \\
&= [(x - 2) + 3i][(x - 2) - 3i] \\
&= (x - 2)^2 - (3i)^2 \\
&= x^2 - 4x + 4 - 9i^2 \\
&= x^2 - 4x + 4 + 9 \\
&= x^2 - 4x + 13
\end{aligned}$$

We can only use synthetic division with linear divisors, so we will need to divide $5x^3 - 19x^2 + 61x + 13$ by $x^2 - 4x + 13$ using long division.

$$\begin{array}{r|l}
 & 5x + 1 \\
\hline
x^2 - 4x + 13 & 5x^3 - 19x^2 + 61x + 13 \\
 & \underline{5x^3 - 20x^2 + 65x} \\
 & \qquad\quad x^2 - 4x + 13 \\
 & \qquad\quad \underline{x^2 - 4x + 13} \\
 & \qquad\qquad\qquad 0
\end{array}$$

Thus, $5x^3 - 19x^2 + 61x + 13$ also has a linear factor of $5x + 1$ and therefore has a zero of $x = -\frac{1}{5}$.

The zeros of the cubic polynomial are:
$x = 2 - 3i$, $x = 2 + 3i$ and $x = -\frac{1}{5}$.

The cubic polynomial in Example 19 had three complex zeros – one real and two imaginary. The quartic polynomial in Example 18 had four complex zeros – two real and two imaginary. In Example 15, we factored a cubic polynomial into a product of three linear polynomials, so the factor theorem says it will have three real zeros. And in the previous section we concluded that, provided we take into account the multiplicity of a zero (e.g. double root), all quadratic polynomials have two complex zeros – either two real zeros or two imaginary zeros. These examples are illustrations of the following useful fact.

Hint: Although for this course we restrict our study to polynomials with real coefficients, it is worthwhile to note that the statement about the number of complex zeros that exist for a polynomial of degree n also holds true for a polynomial with imaginary coefficients. For example, the 2nd degree polynomial $2ix^2 + 4$ has zeros of $1 + i$ and $-1 - i$ (verify this). Note that these two imaginary zeros are not conjugates. Only if a polynomial's coefficients are real must its imaginary zeros occur in conjugate pairs.

Zeros of polynomials of degree *n*

A polynomial of degree $n > 0$ with complex coefficients has exactly n complex zeros, provided that each zero is counted as many times as its multiplicity.

Since imaginary zeros always exist in conjugate pairs then if a polynomial with real coefficients has any imaginary zeros there can only be an even number of them. It logically follows then that a polynomial with an odd degree has at least one real zero. One consequence of this fact is that the graph of an odd-degree polynomial function must intersect the x-axis at least once. This agrees with our claim in Section 3.1 that the end behaviour of a polynomial function is influenced by its degree. Odd-degree polynomial functions will rise as $x \to \infty$ and fall as $x \to -\infty$ (or the other way around if the leading coefficient is negative) producing the same general ∕∖∕ shape as $y = x^3$, and hence will cross the x-axis at least once.

Example 20

Given that $2x + 1$ is a factor of the cubic function $f(x) = 2x^3 - 15x^2 + 24x + 16$

a) completely factorize the polynomial

b) find all of the zeros and their multiplicities

c) sketch its graph for the interval $-1 \leqslant x \leqslant 6$, given that the graph of the function has a turning point at $x = 1$

Solution

a) Remember that synthetic division can only be used for linear divisors of the form $x - c$. Because $2x + 1 = 2\left(x + \frac{1}{2}\right)$, then if $2x + 1$ is a factor $x + \frac{1}{2}$ is also a factor. So we can set up synthetic division with a divisor of $x + \frac{1}{2}$, but we must take the following into account.

$$2x^3 - 15x^2 + 24x + 16 = (2x + 1)\cdot Q(x)$$
$$= 2\left(x + \tfrac{1}{2}\right)\cdot Q(x)$$
$$= \left(x + \tfrac{1}{2}\right)\cdot 2Q(x)$$
$$\frac{2x^3 - 15x^2 + 24x + 16}{x + \frac{1}{2}} = 2Q(x)$$

When the polynomial is divided by $x + \frac{1}{2}$, the quotient will be two times the quotient from dividing by $2x + 1$. Dividing by two will give us the quotient that we want.

$$\begin{array}{r|rrrr} -\frac{1}{2} & 2 & -15 & 24 & 16 \\ & & -1 & 8 & -16 \\ \hline & 2 & -16 & 32 & 0 \end{array}$$

Hence, $2x^3 - 15x^2 + 24x + 16 = \left(x + \frac{1}{2}\right)(2x^2 - 16x + 32)$

and $2x^3 - 15x^2 + 24x + 16 = 2\left(x + \frac{1}{2}\right)\frac{1}{2}(2x^2 - 16x + 32)$

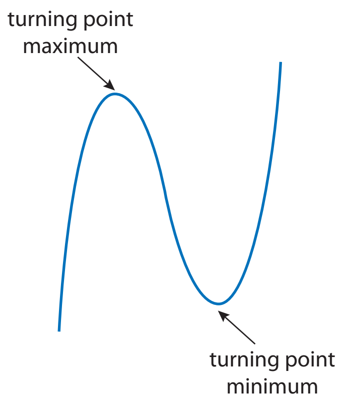
$= (2x + 1)(x^2 - 8x + 16)$ Factorize the quadratic factor.

$= (2x + 1)(x - 4)(x - 4)$ $x^2 - 8x + 16$ fits the pattern $x^2 + 2ax + a^2 = (x + a)^2$

$= (2x + 1)(x - 4)^2$

b) The zeros of $2x^3 - 15x^2 + 24x + 16$ are $x - \frac{1}{2}$ and $x = 4$ (multiplicity of two).

c) Because the polynomial is of degree 3 and its leading coefficient is positive, the end behaviour of the graph will be such that the graph rises as $x \rightarrow \infty$ and falls as $x \rightarrow -\infty$. That means the general shape of the graph will be a ∿ shape with one maximum and one minimum as shown right.

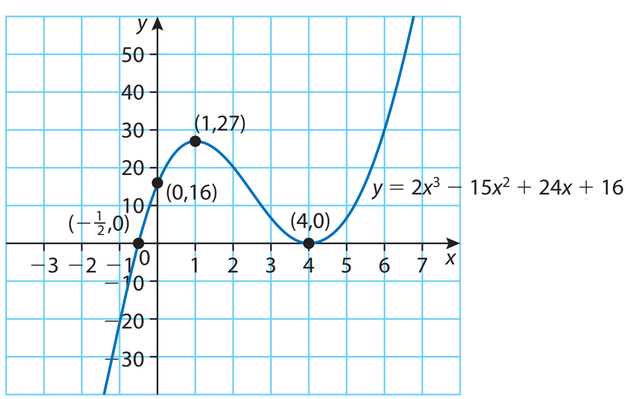


Find the coordinates of the given turning point by evaluating $f(1)$ using synthetic substitution.

$$\begin{array}{r|rrrr} 1 & 2 & -15 & 24 & 16 \\ & & 2 & -13 & 11 \\ \hline & 2 & -13 & 11 & 27 \end{array}$$

$\Rightarrow f(1) = 27$. Hence, the point $(1, 27)$ is on the graph.

Since $f(0) = 16$ then the y-intercept is $(0, 16)$, which means that $(1, 27)$ is a maximum point. Because the zero $x = 4$ has a multiplicity of two, then we know from the previous chapter on quadratic functions that the graph will be tangent to the x-axis at the point $(4, 0)$. The other x-intercept is $\left(-\frac{1}{2}, 0\right)$. We can now make a very accurate sketch of the function.



We know how to find the exact zeros of linear and quadratic functions. The quadratic formula is a general rule that gives the *exact* values of *all* complex zeros of *any* quadratic polynomial using radicals and the coefficients of the polynomial. We also know how to use our GDC to approximate real zeros. In this chapter, we have gained techniques to search for, or verify, the zeros of polynomial functions of degree 3 or higher. This leads us to an important question: Can we find exact values of all complex zeros of any polynomial function of 3rd degree and higher? This question was answered for cubic and quartic polynomials in the 16th century when the Italian mathematician Girolamo Cardano (1501–1576) presented a 'cubic formula' and a 'quartic formula'. These formulae were methods for finding all complex zeros of 3rd degree and 4th degree polynomials using only radicals and coefficients. Cardano's presentation of the formulae depended heavily on the work of other Italian mathematicians. Scipione del Ferro (1465–1526) is given credit as the first to find a general algebraic solution to cubic equations. Cardano's method of solving any cubic was obtained from Niccolo Fontana (1500–1557) known as 'Tartaglia'. Similarly, Cardano solved quartic equations using a method that he learned from his own student Lodovico Ferrari (1522–1565). The methods for solving cubic and quartic equations are quite complicated and are not part of this course. The question of finding formulae for exact zeros of polynomials of degree 5 (quintic) and higher was not resolved until the early 19th century. In 1824, a young Norwegian mathematician, Niels Henrik Abel (1802–1829), proved that it was impossible to find an algebraic formula for a general quintic equation. An even more remarkable discovery was made by the French mathematician Evariste Galois (1811–1832) who died in a pistol duel before turning 21. Galois proved that for any polynomial of degree 5 or greater, it is not possible, except in special cases, to find the exact zeros by using only radicals and the polynomial's coefficients. Mathematicians have developed sophisticated methods of approximating the zeros of polynomial equations of high degree and other types of equations for which there are no algebraic solution methods. These are studied in a branch of advanced mathematics called **numerical analysis**.

Example 21

Find a polynomial P with integer coefficients of least degree having zeros of $x = 2$, $x = -\frac{1}{3}$ and $x = 1 - i$.

Solution

Given that $1 - i$ is a zero then its conjugate $1 + i$ must also be a zero. Thus, the required polynomial has four complex zeros, and four corresponding factors. The four factors are:

$x - 2,\ x + \frac{1}{3},\ x - (1 - i)$ and $x - (1 + i)$

$$\begin{aligned}
P(x) &= (x-2)\left(x+\tfrac{1}{3}\right)[x-(1-i)][x-(1+i)] \\
&= \left(x^2-\tfrac{5}{3}x-\tfrac{2}{3}\right)[(x-1)+i][(x-1)-i] && \text{Multiplying by 3 does not change the zeros …} \\
&= (3x^2-5x-2)[(x-1)^2-i^2] && \text{… but does guarantee integer coefficients.} \\
&= (3x^2-5x-2)(x^2-2x+1+1) \\
&= (3x^2-5x-2)(x^2-2x+2) \\
&= 3x^4-6x^3+6x^2-5x^3+10x^2-10x-2x^2+4x-4
\end{aligned}$$

$$P(x) = 3x^4 - 11x^3 + 14x^2 - 6x - 4$$

There is a theorem called the **fundamental theorem of algebra** that guarantees that *every* polynomial function of non-zero degree with complex coefficients has at least one complex zero. The theorem was first proved by the famous German mathematician Carl Friedrich Gauss (1777–1855). Many of the results in this section on the zeros of polynomials are directly connected with this important theorem.

Sum and product of the roots of any polynomial equation

In the previous section, we found a way to express the sum and product of the roots of a quadratic equation, $ax^2 + bx + c = 0$, in terms of a, b and c. It is natural to wonder whether a similar method could be found for polynomial equations of degree greater than two.

Using the same approach as in the previous section for quadratic equations, let's consider the general cubic equation $ax^3 + bx^2 + cx + d = 0$ whose roots are $x = \alpha$, $x = \beta$ and $x = \gamma$. It follows that this general cubic equation can be written in the form $x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$. Applying the Factor Theorem, it can also be written in the form $(x - \alpha)(x - \beta)(x - \gamma) = 0$. Expanding the brackets gives:

$$\begin{aligned}(x-\alpha)(x-\beta)(x-\gamma) &= x^3 - \alpha x^2 - \beta x^2 - \gamma x^2 + \alpha\beta x + \beta\gamma x + \alpha\gamma x \\ &\quad - \alpha\beta\gamma \\ &= 0\end{aligned}$$

$$x^3 - (\alpha + \beta + \gamma)\, x^2 + (\alpha\beta + \beta\gamma + \alpha\gamma)x - \alpha\beta\gamma = 0$$

Equating coefficients for $x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$ and $x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \alpha\gamma)x - \alpha\beta\gamma = 0$ gives us the following results for the sum and product of the roots for any cubic equation.

$$\alpha + \beta + \gamma = -\frac{b}{a} \text{ and } \alpha\beta\gamma = -\frac{d}{a}$$

This result for the sum and product of the roots of any cubic equation looks very similar to that for any quadratic equation. The only difference is that the product of the roots, $\alpha\beta\gamma$, is the opposite of the quotient $\frac{\text{constant term}}{\text{leading coefficient}}$.

For the general quartic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ with roots α, β, γ and δ, the factored form of the equation expands as follows:

$$\begin{aligned}&(x-\alpha)(x-\beta)(x-\gamma)(x-\delta) = \\ &x^4 - (\alpha+\beta+\gamma+\delta)x^2 + (\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)x - \\ &(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta) + \alpha\beta\gamma\delta = 0\end{aligned}$$

Since this is equivalent to $x^4 + \frac{b}{a}x^3 + \frac{c}{a}x^2 + \frac{d}{a}x + \frac{e}{a} = 0$, then the sum and product of the roots for any quartic equation are:

$$\alpha + \beta + \gamma + \delta = -\frac{b}{a} \text{ and } \alpha\beta\gamma\delta = \frac{e}{a}.$$

These results for the sum and product of roots for polynomial equations of degree 2 (quadratic), degree 3 (cubic) and degree 4 (quartic) lead to the following result for any polynomial function of degree n that we state without a formal proof.

Sum and product of the roots (zeros) of any polynomial equation

For the **polynomial equation of degree *n*** given by $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$, $a_n \neq 0$ the **sum of the roots** is $-\frac{a_{n-1}}{a_n}$ and the **product of the roots** is $\frac{(-1)^n a_0}{a_n}$.

Example 22

Two of the roots of the equation $x^3 - 3x^2 + kx + 75 = 0$ are opposites. Find the values of all the roots and the constant k.

Solution

Let the three unknown roots be represented by α, $-\alpha$ and β.

Then $\alpha - \alpha + \beta = 3 \Rightarrow \beta = 3$ and $\alpha(-\alpha)\beta = -75 \Rightarrow \alpha(-\alpha)(3) = -75 \Rightarrow -3\alpha^2 = -75 \Rightarrow \alpha^2 = 25 \Rightarrow \alpha = \pm 5$

Therefore, the three roots are 5, -5 and 3.

To find the value of k, write the cubic in factored form and expand.

$$\begin{aligned}(x-3)(x+5)(x-5) = 0 &\Rightarrow (x-3)(x^2-25) = 0\\ &\Rightarrow x^3 - 3x^2 - 25x + 75 = 0\end{aligned}$$

Therefore, $k = -25$.

Example 23

Consider the equation $2x^4 - x^3 - 4x^2 + 10x - 4 = 0$. Given that one of the zeros of the equation is $r_1 = 1 + i$, find the other three zeros r_2, r_3 and r_4.

Solution

There are other strategies (e.g. using factors and polynomial division) but it is more efficient to apply what we know about the sum and product of the roots (zeros) of a polynomial equation.

Firstly, since $r_1 = 1 + i$ is a zero, then its conjugate must also be a zero; hence $r_2 = 1 - i$.

From the fact that the sum of the roots is $-\frac{a_{n-1}}{a_n}$, then $r_1 + r_2 + r_3 + r_4 = -\frac{a_3}{a_4}$.

Substituting in known values gives $1 + i + 1 - i + r_3 + r_4 = -\frac{-1}{2}$

$\Rightarrow 2 + r_3 + r_4 = \frac{1}{2} \Rightarrow r_3 + r_4 = -\frac{3}{2}$

Also, since the product of the roots is $\frac{(-1)^n a_0}{a_n}$, then $r_1 r_2 r_3 r_4 = \frac{(-1)^n a_0}{a_n}$.

Substituting gives:

$$\begin{aligned}(1+i)(1-i)r_3 r_4 = \frac{(-1)^4(-4)}{2} &\Rightarrow (1-i^2)r_3 r_4 = -2\\ &\Rightarrow 2r_3 r_4 = -2\\ &\Rightarrow r_3 r_4 = -1\end{aligned}$$

To find r_3 and r_3, we need to use the pair of equations $\begin{cases} r_3 + r_4 = -\frac{3}{2} \\ r_3 r_4 = -1 \end{cases}$

Solving for r_3 in the first equation gives $r_3 = -r_4 - \frac{3}{2}$.

Substituting into the other equation gives: $\left(-r_4 - \frac{3}{2}\right) r_4 = -1$

$$\begin{aligned}&\Rightarrow r_4^2 + \frac{3}{2}r_4 - 1 = 0\\ &\Rightarrow 2r_4^2 + 3r_4 - 2 = 0\\ &\Rightarrow (2r_4 - 1)(r_4 + 2) = 0\\ &\Rightarrow r_4 = \frac{1}{2} \text{ or } r_4 = -2\end{aligned}$$

If $r_4 = \frac{1}{2}$, then $r_3 = -\frac{1}{2} - \frac{3}{2} = -2$. $\left[\text{And if } r_4 = -2\text{, then } r_3 = \frac{1}{2}\right]$

Therefore the other three zeros are $1 - i, \frac{1}{2}$ and -2.

Exercise 3.3

In questions 1–5, two polynomials P and D are given. Use either synthetic division or long division to divide $P(x)$ by $D(x)$, and express $P(x)$ in the form $P(x) = D(x) \cdot Q(x) + R(x)$.

1 $P(x) = 3x^2 + 5x - 5,\ D(x) = x + 3$

2 $P(x) = 3x^4 - 8x^3 + 9x + 5,\ D(x) = x - 2$

3 $P(x) = x^3 - 5x^2 + 3x - 7,\ D(x) = x - 4$

4 $P(x) = 9x^3 + 12x^2 - 5x + 1,\ D(x) = 3x - 1$

5 $P(x) = x^5 + x^4 - 8x^3 + x + 2,\ D(x) = x^2 + x - 7$

6 Given that $x - 1$ is a factor of the function $f(x) = 2x^3 - 17x^2 + 22x - 7$ factorize f completely.

7 Given that $2x + 1$ is a factor of the function $f(x) = 6x^3 - 5x^2 - 12x - 4$ factorize f completely.

8 Given that $x + \frac{2}{3}$ is a factor of the function $f(x) = 3x^4 + 2x^3 - 36x^2 + 24x + 32$ factorize f completely.

In questions 9–12, find the quotient and the remainder.

9 $\dfrac{x^2 - 5x + 4}{x - 3}$

10 $\dfrac{x^3 + 2x^2 + 2x + 1}{x + 2}$

11 $\dfrac{9x^2 - x + 5}{3x^2 - 7x}$

12 $\dfrac{x^5 + 3x^3 - 6}{x - 1}$

In questions 13–16, use synthetic division and the remainder theorem to evaluate $P(c)$.

13 $P(x) = 2x^3 - 3x^2 + 4x - 7, c = 2$

14 $P(x) = x^5 - 2x^4 + 3x^2 + 20x + 3, c = -1$

15 $P(x) = 5x^4 + 30x^3 - 40x^2 + 36x + 14, c = -7$

16 $P(x) = x^3 - x + 1, c = \frac{1}{4}$

17 Given that $x = -6$ is a zero of the polynomial $x^3 + 2x^2 - 19x + 30$ find all remaining zeros of the polynomial.

18 Given that $x = 2$ is a double root of the polynomial $x^4 - 5x^3 + 7x^2 - 4$ find all remaining zeros of the polynomial.

19 Find the values of k such that -3 is a zero of $f(x) = x^3 - x^2 - k^2x$.

20 Find the values of a and b such that 1 and 4 are zeros of $f(x) = 2x^4 - 5x^3 - 14x^2 + ax + b$.

In questions 21–23, find a polynomial with real coefficients satisfying the given conditions.

21 Degree of 3; and zeros of -2, 1 and 4

22 Degree of 4; and zeros of -1, 3 (multiplicity of 2) and -2

23 Degree of 3; and 2 is the only zero (multiplicity of 3)

In questions 24–26, find a polynomial of lowest degree with real coefficients and the given zeros.

24 $x = -1$ and $x = 1 - i$

25 $x = 2$, $x = -4$ and $x = -3i$

26 $x = 3 + i$ and $x = 1 - 2i$

27 Given that $x = 2 - 3i$ is a zero of $f(x) = x^3 - 7x^2 + 25x - 39$ find the other remaining zeros.

28 The polynomial $6x^3 + 7x^2 + ax + b$ has a remainder of 72 when divided by $x - 2$ and is exactly divisible (i.e. remainder is zero) by $x + 1$.

a) Calculate a and b.

b) Show that $2x - 1$ is also a factor of the polynomial and, hence, find the third factor.

29 The polynomial $p(x) = (ax + b)^3$ leaves a remainder of -1 when divided by $x + 1$, and a remainder of 27 when divided by $x - 2$. Find the values of the real numbers a and b.

30 The quadratic polynomial $x^2 - 2x - 3$ is a factor of the quartic polynomial function $f(x) = 4x^4 - 6x^3 - 15x^2 - 8x - 3$. Find all of the zeros of the function f. Express the zeros exactly and completely simplified.

31 $x - 2$ and $x + 2$ are factors of $x^3 + ax^2 + bx + c$, and it leaves a remainder of 10 when divided by $x - 3$. Find the values of a, b and c.

32 Let $P(x) = x^3 + px^2 + qx + r$. Two of the zeros of $P(x) = 0$ are 3 and $1 + 4i$. Find the value of p, q and r.

33 When divided by $(x + 2)$ the expression $5x^3 - 3x^2 + ax + 7$ leaves a remainder of R. When the expression $4x^3 + ax^2 + 7x - 4$ is divided by $(x + 2)$ there is a remainder of $2R$. Find the value of the constant a.

34 The polynomial $x^3 + mx^2 + nx - 8$ is divisible by $(x + 1 + i)$. Find the value of m and n.

35 Given that the roots of the equation $x^3 - 9x^2 + bx - 216 = 0$ are consecutive terms in a geometric sequence, find the value of b and solve the equation.

36 a) Prove that when a polynomial $P(x)$ is divided by $ax - b$ the remainder is $P\left(\frac{b}{a}\right)$.

b) Hence, find the remainder when $9x^3 - x + 5$ is divided by $3x + 2$.

37 Find the sum and product of the roots of the following equations.

a) $x^4 - \frac{2}{3}x^3 + 3x^2 - 2x + 5 = 0$

b) $(x - 2)^3 = x^4 - 1$

c) $\dfrac{3}{x^2 + 2} = \dfrac{2x^2 - x}{2x^5 + 1}$

38 If α, β and γ are the three roots of the cubic equation $ax^3 + bx^2 + cx + d = 0$, show that $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$.

39 One of the zeros of the equation $x^3 - 63x + 162 = 0$ is double another zero. Find all three zeros.

40 Find the three zeros of the equation $x^3 - 6x^2 - 24x + 64 = 0$ given that they are consecutive terms in a geometric sequence. [Hint: let the zeros be represented by $\frac{\alpha}{r}$, α, αr where r is the common ratio.]

41 Consider the equation $x^5 - 12x^4 + 62x^3 - 166x^2 + 229x - 130 = 0$. Given that two of the zeros of the equation are $x = 3 - 2i$ and $x = 2$, find the remaining three zeros.

42 Find the value of k such that the zeros of the equation $x^3 - 6x^2 + kx + 10 = 0$ are in arithmetic progression, that is, they can be represented by α, $\alpha + d$ and $\alpha + 2d$ for some constant d. [Hint: use the result from question 38.]

43 Find the value of k if the roots of the equation $x^3 + 3x^2 - 6x + k = 0$ are in geometric progression.

3.4 Rational functions

Another important category of algebraic functions is rational functions, which are functions in the form $R(x) = \frac{f(x)}{g(x)}$ where f and g are polynomials and the domain of the function R is the set of all real numbers except the real zeros of polynomial g in the denominator. Some examples of rational functions are

$$p(x) = \frac{1}{x-5}, \quad q(x) = \frac{x+2}{(x+3)(x-1)}, \quad \text{and} \quad r(x) = \frac{x}{x^2+1}$$

The domain of p excludes $x = 5$, and the domain of q excludes $x = -3$ and $x = 1$. The domain of r is all real numbers because the polynomial $x^2 + 1$ has no real zeros.

Example 24

Find the domain and range of $h(x) = \frac{1}{x-2}$. Sketch the graph of h.

Solution

Because the denominator is zero when $x = 2$, the domain of h is all real numbers except $x = 2$, i.e. $x \in \mathbb{R}, x \neq 2$. Determining the range of the function is a little less straightforward. It is clear that the function could never take on a value of zero because that will only occur if the numerator is zero. And since the denominator can have any value except zero it seems that the function values of h could be any real number except zero. To confirm this and to determine the behaviour of the function (and shape of the graph), some values of the domain and range (pairs of coordinates) are displayed in the tables below.

● **Hint:** A fraction is only zero if its numerator is zero.

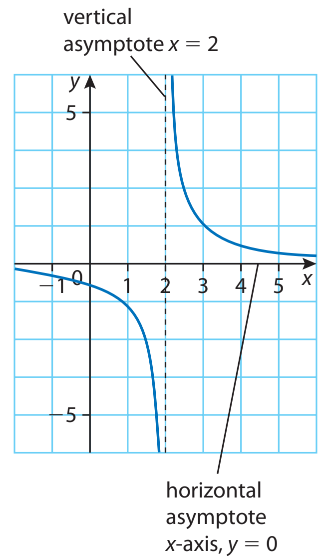
x approaches 2 from the left

x	$h(x)$
−98	−0.01
−8	−0.1
0	−0.5
1	−1
1.5	−2
1.9	−10
1.99	−100
1.999	−1000

x approaches 2 from the right

x	$h(x)$
102	0.01
12	0.1
4	0.5
3	1
2.5	2
2.1	10
2.01	100
2.001	1000

The values in the tables provide clear evidence that the range of h is all real numbers except zero, i.e. $h(x) \in \mathbb{R}, h(x) \neq 0$. The values in the tables also show that as $x \to -\infty$, $h(x) \to 0$ from below (sometimes written $h(x) \to 0^-$) and as $x \to +\infty$, $h(x) \to 0$ from above ($h(x) \to 0^+$). It follows that the line with equation $y = 0$ (the x-axis) is a horizontal asymptote for the graph of h. As $x \to 2$ from the left (sometimes written $x \to 2^-$), $h(x)$ appears to decrease without bound, whereas as $x \to 2$ from the right ($x \to 2^+$), $h(x)$ appears to increase without bound. This indicates that the graph of h will have a vertical asymptote at $x = 2$. This behaviour is confirmed by the graph at left.



Horizontal and vertical asymptotes

The line $y = c$ is a **horizontal asymptote** of the graph of the function f if at least one of the following statements is true:

- as $x \to +\infty$, then $f(x) \to c^+$
- as $x \to -\infty$, then $f(x) \to c^+$
- as $x \to +\infty$, then $f(x) \to c^-$
- as $x \to -\infty$, then $f(x) \to c^-$

The line $x = d$ is a **vertical asymptote** of the graph of the function f if at least one of the following statements is true:

- as $x \to d^+$, then $f(x) \to +\infty$
- as $x \to d^+$, then $f(x) \to -\infty$
- as $x \to d^-$, then $f(x) \to +\infty$
- as $x \to d^-$, then $f(x) \to -\infty$

Example 25

Consider the function $f(x) = \dfrac{3x^2 - 12}{x^2 + 3x - 4}$. Sketch the graph of f and identify any asymptotes and any x- or y-intercepts. Use the sketch to confirm the domain and range of the function.

Solution

Firstly, let's completely factorize both the numerator and denominator.

$$f(x) = \frac{3x^2 - 12}{x^2 + 3x - 4} = \frac{3(x+2)(x-2)}{(x-1)(x+4)}$$

Axis intercepts:

The x-intercepts will occur where the numerator is zero. Hence, the x-intercepts are $(-2, 0)$ and $(2, 0)$. A y-intercept will occur when $x = 0$.

$f(0) = \dfrac{3(2)(-2)}{(-1)(4)} = 3$, so the y-intercept is $(0, 3)$.

Vertical asymptote(s):

Any vertical asymptote will occur where the denominator is zero, that is, where the function is undefined. From the factored form of f we see that the vertical asymptotes are $x = 1$ and $x = -4$. We need to determine if the graph of f falls ($f(x) \to -\infty$) or rises ($f(x) \to \infty$) on either side of each vertical asymptote. It's easiest to do this by simply analyzing what the sign of h will be as x approaches 1 and -4 from both the left and right. For example, as $x \to 1^-$ we can use a test value close to and to the left of 1 (e.g. $x = 0.9$) to check whether $f(x)$ is positive or negative to the left of 1.

$$f(x) = \frac{3(0.9+2)(0.9-2)}{(0.9-1)(0.9+4)} \Rightarrow \frac{(+)(-)}{(-)(+)} \Rightarrow f(x) > 0 \Rightarrow \text{as } x \to 1^-,$$

then $f(x) \to +\infty$ (rises)

As $x \to 1^+$ we use a test value close to and to the right of 1 (e.g. $x = 1.1$) to check whether $f(x)$ is positive or negative to the right of 1.

● **Hint:** The farther the number n is from 0, the closer the number $\frac{1}{n}$ is to 0. Conversely, the closer the number n is to 0, the farther the number $\frac{1}{n}$ is from 0. These facts can be expressed simply as:

$$\frac{1}{\text{BIG}} = \text{little and } \frac{1}{\text{little}} = \text{BIG}$$

They can also be expressed more mathematically using the concept of a limit expressed in limit notation as: $\lim\limits_{n \to \infty} \frac{1}{n} = 0$ and $\lim\limits_{n \to 0} \frac{1}{n} = \infty$.

Note: Infinity is not a number, so $\lim\limits_{n \to 0} \frac{1}{n}$ actually does not exist, but writing $\lim\limits_{n \to 0} \frac{1}{n} = \infty$ expresses the idea that $\frac{1}{n}$ increases without bound as n approaches 0.

$$f(x) = \frac{3(1.1+2)(1.1-2)}{(1.1-1)(1.1+4)} \Rightarrow \frac{(+)(-)}{(+)(+)} \Rightarrow f(x) < 0 \Rightarrow \text{as } x \to 1^+,$$

then $f(x) \to -\infty$ (falls)

Conducting similar analysis for the vertical asymptote of $x = -4$, produces:

$$f(x) = \frac{3(-4.1+2)(-4.1-2)}{(-4.1-1)(-4.1+4)} \Rightarrow \frac{(-)(-)}{(-)(-)} \Rightarrow f(x) > 0 \Rightarrow \text{as } x \to 4^-,$$

then $f(x) \to +\infty$ (rises)

$$f(x) = \frac{3(-3.9+2)(-3.9-2)}{(-3.9-1)(-3.9+4)} \Rightarrow \frac{(-)(-)}{(-)(+)} \Rightarrow f(x) < 0 \Rightarrow \text{as } x \to 4^+,$$

then $f(x) \to -\infty$ (falls)

Horizontal asymptote(s):

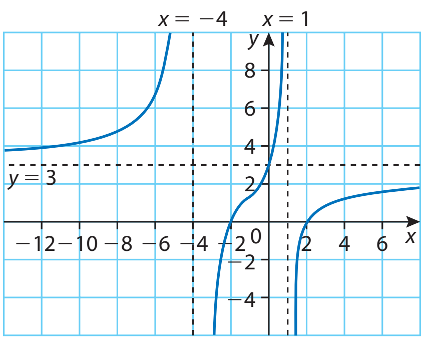
A horizontal asymptote (if it exists) is the value that $f(x)$ approaches as $x \to \pm\infty$. To find this value, we divide both the numerator and denominator by the highest power of x that appears in the denominator (x^2 for function f).

$$f(x) = \frac{\frac{3x^2}{x^2} - \frac{12}{x^2}}{\frac{x^2}{x^2} + \frac{3x}{x^2} - \frac{4}{x^2}} \text{ then, as } x \to \pm\infty, f(x) = \frac{3-0}{1+0-0} = 3$$

Hence, the horizontal asymptote is $y = 3$.

Sketch of graph:

Now we know the behaviour (rising or falling) of the function on either side of each vertical asymptote and that the graph will approach the horizontal asymptote as $x \to \pm\infty$, an accurate sketch of the graph can be made as shown right.



Domain and range:

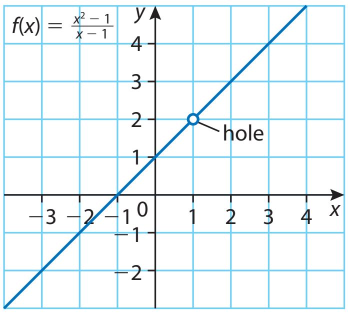
Because the zeros of the polynomial in the denominator are $x = 1$ and $x = -4$, the domain of f is all real numbers except 1 and -4.
From our analysis and from the sketch of the graph, it is clear that between $x = -4$ and $x = 1$ the function takes on all values from $-\infty$ to $+\infty$, therefore the range of f is all real numbers.

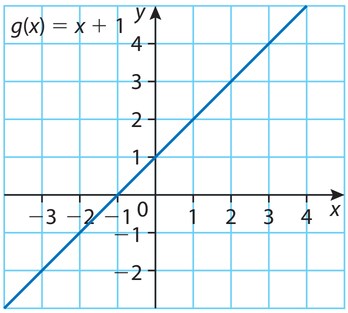
We are in the habit of cancelling factors in algebraic expressions (Section 1.5), such as

$$\frac{x^2-1}{x-1} = \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}} = x+1$$

However, the function $f(x) = \frac{x^2-1}{x-1}$ and the function $g(x) = x + 1$ are **not** the same function. The difference occurs when $x = 1$.

$f(1) = \frac{1^2-1}{1-1} = \frac{0}{0}$, which is undefined, and $g(1) = 1 + 1 = 2$. So, 1 is not in the domain of f but it is in the domain of g. As we might expect the graphs of the two functions appear identical, but upon closer inspection it is clear that there is a 'hole' in the graph of f at the point $(1, 2)$. Thus, f is a *discontinuous* function but the polynomial function g is continuous. f and g are different functions.





● **Hint:** Try graphing $\dfrac{x^2-1}{x-1}$ on your GDC and zooming in closely to the region around the point $(1, 2)$. Can you see the 'hole'?

In working with rational functions, we often assume that every linear factor that appears in both the numerator and in the denominator has been cancelled. Therefore, for a rational function in the form $\dfrac{f(x)}{g(x)}$, we can usually assume that the polynomial functions f and g have no common factors.

Example 26

Find any asymptotes for the function $p(x) = \dfrac{x^2-9}{x-4}$.

Solution

The denominator is zero when $x = 4$, thus the line with equation $x = 4$ is a vertical asymptote. Although the numerator $x^2 - 9$ is not divisible by $x - 4$, it does have a larger degree. Some insight into the behaviour of function p may be gained by dividing $x - 4$ into $x^2 - 9$. Since the degree of the numerator is one greater than the degree of the denominator, the quotient will be a linear polynomial. Recalling from the previous section that $\dfrac{P(x)}{D(x)} = Q(x) + \dfrac{R(x)}{D(x)}$, where Q and R are the quotient and remainder, we can rewrite $p(x)$ as a linear polynomial plus a fraction.

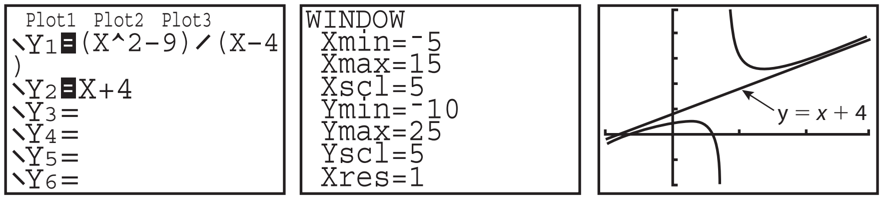
Since the denominator is in the form $x - c$ we can carry out the division efficiently by means of synthetic division.

$$\begin{array}{c|ccc} 4 & 1 & 0 & -9 \\ & & 4 & 16 \\ \hline & 1 & 4 & 7 \end{array}$$

Hence, $p(x) = \dfrac{x^2-9}{x-4} = x + 4 + \dfrac{7}{x-4}$.

As $x \to \pm\infty$, the fraction $\dfrac{7}{x-4} \to 0$. This tells us about the end behaviour of function p, namely that the graph of p will get closer and closer to the line $y = x + 4$ as the values of x get further away from the origin. Symbolically, this can be expressed as follows: as $x \to \pm\infty$, $p(x) \to x + 4$.

We can graph both the rational function $p(x)$ and the line $y = x + 4$ on our GDC to visually confirm our analysis.



If a line is an asymptote of a graph but it is neither horizontal nor vertical, it is called an **oblique asymptote** (sometimes called a slant asymptote).

The graph of any rational function of the form $\frac{f(x)}{g(x)}$, where the degree of function f is one more than the degree of function g will have an oblique asymptote.

Using Example 25 as a model, we can set out a general procedure for analyzing a rational function leading to a sketch of its graph and determining its domain and range.

Analyzing a rational function $R(x) = \frac{f(x)}{g(x)}$ given functions f and g have no common factors

1. Factorize: Completely factorize both the numerator and denominator.
2. Intercepts: A zero of f will be a zero of R and hence an x-intercept of the graph of R. The y-intercept is found by evaluating $R(0)$.
3. Vertical asymptotes: A zero of g will give the location of a vertical asymptote (if any). Then perform a sign analysis to see if $R(x) \to +\infty$ or $R(x) \to -\infty$ on either side of each vertical asymptote.
4. Horizontal asymptote: Find the horizontal asymptote (if any) by dividing both f and g by the highest power of x that appears in g, and then letting $x \to \pm\infty$.
5. Oblique asymptotes: If the degree of f is one more than the degree of g, then the graph of R will have an oblique asymptote. Divide g into f to find the quotient $Q(x)$ and remainder. The oblique asymptote will be the line with equation $y = Q(x)$.
6. Sketch of graph: Start by drawing dashed lines where the asymptotes are located. Use the information about the intercepts, whether $Q(x)$ falls or rises on either side of a vertical asymptote, and additional points as needed to make an accurate sketch.
7. Domain and range: The domain of R will be all real numbers except the zeros of g. You need to study the graph carefully in order to determine the range. Often, but not always (as in Example 25), the value of the function at the horizontal asymptote will not be included in the range.

End behaviour of a rational function

Let R be the rational function given by

$$R(x) = \frac{f(x)}{g(x)} = \frac{a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0}{b_mx^m + b_{m-1}x^{m-1} + \ldots + b_1x + b_0}$$

where functions f and g have no common factors. Then the following holds true:

1. If $n < m$, then the x-axis (line $y = 0$) is a horizontal asymptote for the graph of R.
2. If $n = m$, then the line $y = \frac{a_n}{b_m}$ is a horizontal asymptote for the graph of R.
3. If $n > m$, then the graph of R has no horizontal asymptote. However, if the degree of f is one more than the degree of g, then the graph of R will have an oblique asymptote.

Exercise 3.4

In questions 1–10, sketch the graph of the rational function without the aid of your GDC. On your sketch clearly indicate any x- or y-intercepts and any asymptotes (vertical, horizontal or oblique). Use your GDC to verify your sketch.

1 $f(x) = \dfrac{1}{x+2}$

2 $g(x) = \dfrac{3}{x-2}$

3 $h(x) = \dfrac{1-4x}{1-x}$

4 $R(x) = \dfrac{x}{x^2-9}$

5 $p(x) = \dfrac{2}{x^2+2x-3}$

6 $M(x) = \dfrac{x^2+1}{x}$

7 $f(x) = \dfrac{x}{x^2+4x+4}$

8 $h(x) = \dfrac{x^2+2x}{x-1}$

9 $g(x) = \dfrac{2x+8}{x^2-x-12}$

10 $C(x) = \dfrac{x-2}{x^2-4x}$

In questions 11–14, use your GDC to sketch a graph of the function, and state the domain and range of the function.

11 $f(x) = \dfrac{2x^2+5}{x^2-4}$

12 $g(x) = \dfrac{x+4}{x^2+3x-4}$

13 $h(x) = \dfrac{6}{x^2+6}$

14 $r(x) = \dfrac{x^2-2x+1}{x-1}$

In questions 15–18, use your GDC to sketch a graph of the function. Clearly label any x- or y-intercepts and any asymptotes.

15 $f(x) = \dfrac{2x-5}{2x^2+9x-18}$

16 $g(x) = \dfrac{x^2+x+1}{x-1}$

17 $h(x) = \dfrac{3x^2}{x^2+x+2}$

18 $g(x) = \dfrac{1}{x^3-x^2-4x+4}$

19 If a, b and c are all positive, sketch the curve $y = \dfrac{x-a}{(x-b)(x-c)}$ for each of the following conditions:

a) $a < b < c$ b) $b < a < c$ c) $b < c < a$

20 A drug is given to a patient and the concentration of the drug in the bloodstream is carefully monitored. At time $t \geqslant 0$ (in minutes after patient receiving the drug), the concentration, in milligrams per litre (mg/l) is given by the following function.

$$C(t) = \frac{25t}{t^2+4}$$

a) Sketch a graph of the drug concentration (mg/l) versus time (min).

b) When does the highest concentration of the drug occur, and what is it?

c) What eventually happens to the concentration of the drug in the bloodstream?

d) How long does it take for the concentration to drop below 0.5 mg/l?

3.5 Other equations and inequalities

We have studied some approaches to analyzing and solving polynomial equations in this chapter. Some problems lead to equations with expressions that are not polynomials, for example, expressions with radicals, fractions, or absolute value. Problems in mathematics often do not involve equations but inequalities. We need to be familiar with effective methods for solving inequalities involving polynomials – and again, radicals, fractions, or absolute value.

Equations involving a radical

Example 27 – Solving an equation with a single radical expression

Solve for x: $\sqrt{3x+6} = 2x + 1$

Solution

Squaring both sides gives $3x + 6 = (2x+1)^2$

$$3x + 6 = 4x^2 + 4x + 1$$

$$4x^2 + x - 5 = 0$$

Factorizing: $(4x+5)(x-1) = 0$

$$x = -\tfrac{5}{4} \text{ or } x = 1$$

Check both solutions in the original equation:

When $x = -\frac{5}{4}$, $\sqrt{3\left(-\frac{5}{4}\right) + 6} = 2\left(-\frac{5}{4}\right) + 1 \Rightarrow \sqrt{\frac{9}{4}} = -\frac{3}{2} \Rightarrow \frac{3}{2} \neq -\frac{3}{2}$

Therefore, $x = -\frac{5}{4}$ is *not* a solution.

When $x = 1$, $\sqrt{3(1) + 6} = 2(1) + 1 \Rightarrow \sqrt{9} = 3 \Rightarrow 3 = 3$

Therefore, $x = 1$ is the only solution.

If two quantities are equal, for example $a = b$, then it is certainly true that $a^2 = b^2$, and $a^3 = b^3$, etc. However, the converse is not necessarily true. A simple example can illustrate this.

Consider the trivial equation $x = 3$. There is only one value of x that makes the equation true – and that is 3. Now if we take this original equation and square both sides we transform it to the equation $x^2 = 9$. This transformed equation has two solutions, 3 and -3, so it is not equivalent to the original equation. By squaring both sides we gained an extra solution, often called an **extraneous solution**, that satisfies the transformed equation but not the original equation as occurred in Example 27. Whenever you raise both sides of an equation by a power it is imperative that you check all solutions in the original equation.

Every solution of the equation $a = b$ is also a solution of the equation $a^n = b^n$, but it is not necessarily true that every solution of $a^n = b^n$ is a solution of $a = b$.

Example 28 – Solving an equation with two radical expressions

Solve for x in the equation $\sqrt{2x-3} - \sqrt{x+7} = 2$.

Solution

Squaring both sides of the original equation will produce a messy expression on the left side, so it is better to rearrange the terms so that one side of the equation contains only a single radical term.

$$\sqrt{2x-3} = 2 + \sqrt{x+7}$$

$$(\sqrt{2x-3})^2 = (2 + \sqrt{x+7})^2$$

$$2x - 3 = 4 + 4\sqrt{x+7} + x + 7$$

$$x - 14 = 4\sqrt{x+7}$$

$$(x-14)^2 = (4\sqrt{x+7})^2$$

Squaring both sides again to eliminate the radical.

$$x^2 - 28x + 196 = 16(x+7)$$

$$x^2 - 44x + 84 = 0$$

$$(x-2)(x-42) = 0$$

$$x = 2 \text{ or } x = 42$$

Check both solutions in the original equation:

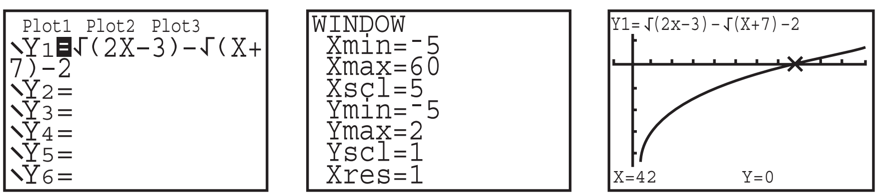
When $x = 2$, $\sqrt{2(2)-3} \stackrel{?}{=} 2 + \sqrt{2+7} \Rightarrow \sqrt{1} \stackrel{?}{=} 2 + \sqrt{9} \Rightarrow 1 \neq 5$

Thus, $x = 2$ is *not* a solution.

When $x = 42$, $\sqrt{2(42)-3} \stackrel{?}{=} 2 + \sqrt{42+7} \Rightarrow \sqrt{81} \stackrel{?}{=} 2 + \sqrt{49} \Rightarrow 9 = 2 + 7$

Thus, $x = 42$ is a solution.

We can verify the single solution of $x = 42$ using our GDC by graphing the equation $y = \sqrt{2x-3} - \sqrt{x+7} - 2$ and looking for x-intercepts (zeros). Since we are restricted to real number solutions then the smallest possible value for x that can be substituted into the equation is $\frac{3}{2}$. This helps determine a suitable viewing window for the graph on our GDC.



This verifies that $x = 42$ is the only solution to the equivalent equation $\sqrt{2x-3} = 2 + \sqrt{x+7}$.

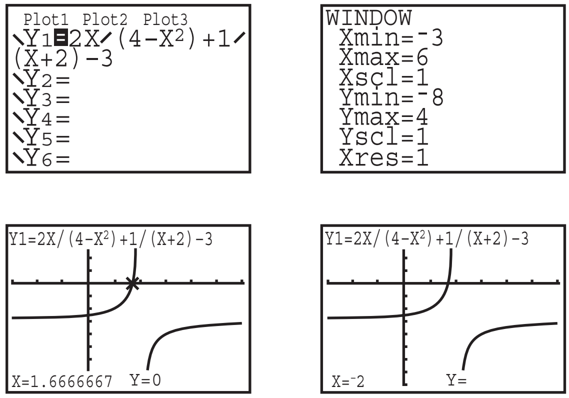
Equations involving fractions

It is also possible for extraneous solutions to appear when solving equations with fractions.

Example 29 – An extraneous root in an equation with fractions

Find all real solutions of the equation $\frac{2x}{4-x^2} + \frac{1}{x+2} = 3$ and verify solution(s) with a GDC.

Solution



Multiply both sides of the equation by the least common denominator of the fractions, $4 - x^2$.

$$\frac{4 - x^2}{1} \cdot \frac{2x}{4 - x^2} + \frac{(2 - x)(2 + x)}{1} \cdot \frac{1}{x + 2} = 3(4 - x^2)$$

Factorizing $4 - x^2$ gives $(2 - x)(2 + x)$.

$$2x + 2 - x = 12 - 3x^2$$

$$3x^2 + x - 10 = 0$$

$$(3x - 5)(x + 2) = 0$$

$$x = \tfrac{5}{3} \text{ or } x = -2$$

Clearly $x = -2$ cannot be a solution because that would cause division by zero in the original equation.

The GDC images show that the equation $y = \dfrac{2x}{4 - x^2} + \dfrac{1}{x + 2} - 3$ has an x-intercept at $\left(\frac{5}{3}, 0\right)$, confirming the solution $x = \frac{5}{3}$.

● **Hint:** Not only is it possible to *gain* an extraneous solution when solving certain equations, it is also possible to *lose* a correct solution by incorrectly dividing both sides of an equation by a common factor. For example, solve for x in the equaton $4(x + 2)^2 = 3x(x + 2)$. Dividing both sides by $(x + 2)$, gives $4(x + 2) = 3x \Rightarrow 4x + 8 = 3x \Rightarrow x = -8$. However, there are two solutions, $x = -8$ *and* $x = -2$. The solution of $x = -2$ was lost because a factor of $x + 2$ was eliminated from both sides of the original equation. This is a common error to be avoided.

Equations in quadratic form

In Section 3.2 we covered methods of solving quadratic equations. As the three previous examples illustrate, quadratic equations commonly appear in a range of mathematical problems. The methods of solving quadratics can sometimes be applied to other equations. An equation in the form $at^2 + bt + c = 0$, where t is an algebraic expression, is an equation in **quadratic form**. We can solve such equations by substituting for the algebraic expression and then apply an appropriate method for solving a quadratic equation.

Example 30 – A 4th degree polynomial equation in quadratic form

Find all real solutions of the equation $2m^4 - 5m^2 + 2 = 0$.

Solution

The equation can be written as $2(m^2)^2 - 5(m^2) + 2 = 0$ showing it is quadratic in terms of m^2. Let $t = m^2$, and substituting gives $2t^2 - 5t + 2 = 0$. Solve for t, substitute m^2 back in for t, and then solve for m.

$2m^4 - 5m^2 + 2 = 0$

Substitute t for m^2 $\quad 2t^2 - 5t + 2 = 0$

$(2t - 1)(t - 2) = 0$

$t = \frac{1}{2}$ or $t = 2$

Substituting m^2 for t $\quad m^2 = \frac{1}{2}$ or $m^2 = 2$

$m = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}$ or $m = \pm\sqrt{2}$

These four solutions – which are two pairs of opposites – can be checked by substituting them directly into the original equation. A value for m will be raised to the 4th and 2nd powers, thus we only need to check one value from each pair of opposites.

When $m = \frac{\sqrt{2}}{2}$, $2\left(\frac{\sqrt{2}}{2}\right)^4 - 5\left(\frac{\sqrt{2}}{2}\right)^2 + 2 = 0 \Rightarrow 2\left(\frac{1}{4}\right) - 5\left(\frac{1}{2}\right) + 2 = 0$

$\Rightarrow \frac{1}{2} - \frac{5}{2} + 2 = 0 \Rightarrow 0 = 0$

When $m = \sqrt{2}$, $2(\sqrt{2})^4 - 5(\sqrt{2})^2 + 2 = 0 \Rightarrow 2(4) - 5(2) + 2 = 0$

$\Rightarrow 8 - 10 + 2 = 0 \Rightarrow 0 = 0$

Therefore, the solutions to the equation are $m = \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, \sqrt{2}$ and $-\sqrt{2}$.

Example 31 – Another equation in quadratic form

Find all solutions, expressed exactly, to the equation $w^{\frac{1}{2}} = 4w^{\frac{1}{4}} - 2$.

Solution

$w^{\frac{1}{2}} - 4w^{\frac{1}{4}} + 2 = 0$ — Set the equation to zero.

$\left(w^{\frac{1}{4}}\right)^2 - 4\left(w^{\frac{1}{4}}\right) + 2 = 0$ — Attempt to write in quadratic form: $at^2 + bt + c = 0$

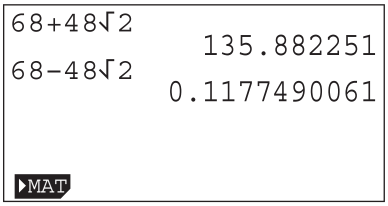
$t^2 - 4t + 2 = 0$ — Make appropriate substitution; in this case, let $w^{\frac{1}{4}} = t$.

$t = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(2)}}{2}$ — Trinomial does not factorize; apply quadratic formula.

$t = \frac{4 \pm \sqrt{8}}{2} = \frac{4 \pm 2\sqrt{2}}{2}$

$t = 2 \pm \sqrt{2}$

$w^{\frac{1}{4}} = 2 \pm \sqrt{2}$ Substituting $w^{\frac{1}{4}}$ back in for t; raise both sides to 4th power.



$w = (2 + \sqrt{2})^4$ or $w = (2 - \sqrt{2})^4$

$w = ((2 + \sqrt{2})^2)^2$ or $w = ((2 - \sqrt{2})^2)^2$

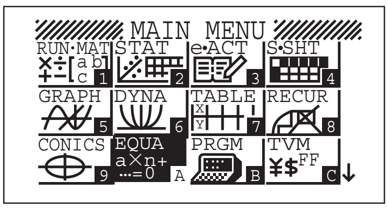
$w = (6 + 4\sqrt{2})^2$ or $w = (6 - 4\sqrt{2})^2$

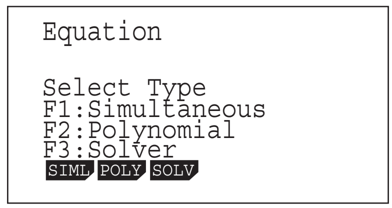
$w = 68 + 48\sqrt{2} \approx 135.882$ or $w = 68 - 48\sqrt{2} \approx 0.117\,749$ (approx. values found with GDC)

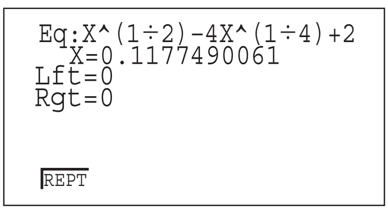
It will be difficult to check these two solutions by substituting them directly into the original equation as we did in the previous example. It will be more efficient to use our GDC.

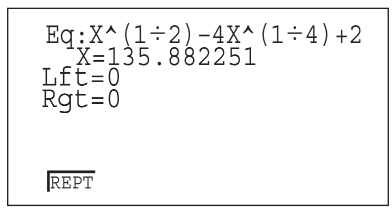
● **Hint:** We will encounter equations in later chapters – for example, equations with logarithms and trigonometric functions – that will be in quadratic form.

Most GDC models have an equation 'solver'. The main limitation of this GDC feature is that it will usually return only approximate solutions. However, even if exact solutions are required, approximate solutions from a GDC are still very helpful as a check of the exact solutions obtained algebraically.









Equations involving absolute value

Equations involving absolute value occur in a range of different topics in mathematics. To solve an equation containing one or more absolute value expressions, we apply the definition from Section 1.1, which states that the absolute value of a real number a, denoted by $|a|$, is given by

$$|a| = \begin{cases} a & \text{if } a \geqslant 0 \\ -a & \text{if } a < 0 \end{cases}$$

Also recall that in Section 1.1 we stated that $|a|$ is the distance between the coordinate a and the origin on the real number line.

Example 32 – Equation with an absolute value expression

Use an algebraic approach to solve the equation $|2x + 7| = 13$. Check any solution(s) on a GDC.

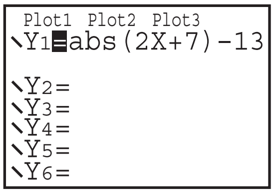
Solution

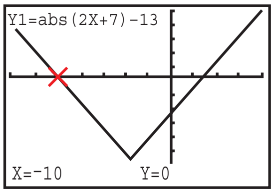
The expression inside the absolute value symbols must be either 13 or -13, so $2x + 7$ equals 13 or -13. Hence, the given equation is satisfied if either

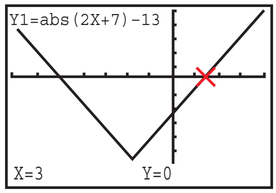
$$\begin{array}{lll} 2x + 7 = 13 & \text{or} & 2x + 7 = -13 \\ 2x = 6 & & 2x = -20 \\ x = 3 & & x = -10 \end{array}$$

The solutions are $x = 3$ and $x = -10$.

To check the solutions on a GDC, graph the equation $y = |2x + 7| - 13$ and confirm that $x = 3$ and $x = -10$ are the x-intercepts of the graph.







The x-intercepts of the graph of $y = |2x + 7| - 13$ agree with the solutions to the equation.

Example 33 – Equation with two absolute value expressions

Find algebraically the solution(s) to the equation $|2x - 1| = |7 - 3x|$. Check the solution(s) graphically.

Solution

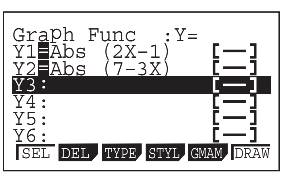
There are four possibilities:

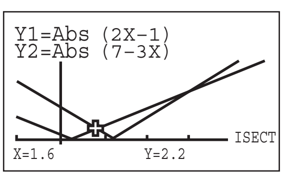
$$2x - 1 = 7 - 3x \text{ or } 2x - 1 = -(7 - 3x) \text{ or } -(2x - 1) = 7 - 3x$$
$$\text{or } -(2x - 1) = -(7 - 3x)$$

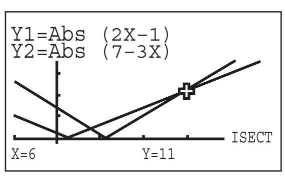
The first and last equations are equivalent, and the second and third equations are also equivalent. Thus, it is only necessary to solve the first two equations.

$$\begin{array}{lll} 2x - 1 = 7 - 3x & \text{or} & 2x - 1 = -(7 - 3x) \\ 5x = 8 & & 2x - 1 = -7 + 3x \\ x = \frac{8}{5} & & 6 = x \Rightarrow x = 6 \end{array}$$

To check, we can graph the equations $y_1 = |2x - 1|$ and $y_2 = |7 - 3x|$, and confirm that the x-coordinates of their points of intersection agree with the solutions to the given equation.







Solving inequalities

Working with inequalities is very important for many of the topics in this course. Inequalities were covered in Section 1.1 in the context of order on the real number line. Recall the four important properties for inequalities.

Properties of inequalities

For three real numbers a, b and c:

1. If $a > b$ and $b > c$, then $a > c$.
2. If $a > b$ and $c > 0$, then $ac > bc$.
3. If $a > b$ and $c < 0$, then $ac < bc$.
4. If $a > b$, then $a + c > b + c$.

Quadratic inequalities

In the topics covered in this course, you will need to be as proficient with solving inequalities as with solving equations. We solved some simple linear inequalities in Section 1.1. Here we will consider strategies for other inequalities – particularly involving quadratic and absolute value expressions.

Example 34 – A quadratic inequality

Find the values of x that solve the inequality $x^2 > x$.

Solution

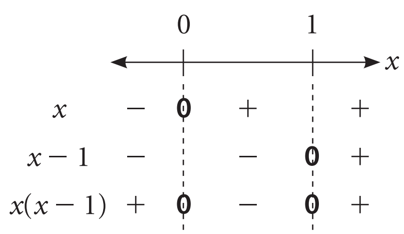
It is possible to determine the solution set to this inequality by a method of trial and error, or simply using a mental process. That may be successful but generally speaking it is a good idea to attempt to find the solution set by some algebraic method and then check, usually by means of a GDC. For this example, it is tempting to consider dividing both sides by x, but that cannot be done because it is not known whether x is positive or negative. Recall that when multiplying or dividing both sides of an inequality by a negative number it is necessary to reverse the inequality sign (3rd property of inequalities listed above). Instead a better approach is to place all terms on one side of the inequality (with zero on the other side) and then try to factorize.

$x^2 > x$

$x^2 - x > 0$

$x(x - 1) > 0$ Now analyze the signs of the two different factors in a 'sign chart'.

sign chart



The sign chart indicates that the product of the two factors, $x(x - 1)$, will be positive when x is less than 0 or greater than 1. Therefore, the solution set is $x < 0$ or $x > 1$.

● **Hint:** The solution set, $x < 0$ or $x > 1$, for Example 34 comprises two intervals that do not intersect (disjoint). It is incorrect to write the solution as $0 > x > 1$, or as $1 < x < 0$. Both of these formats imply that the solution set consists of the values of x *between* 0 and 1, but that is not the case. Only write the 'combined' inequality $a < x < b$ if $x > a$ *and* $x < b$ where the two intervals are intersecting *between* a and b.

Inequalities with quadratic polynomials arise in many different contexts. Problems in which we need to analyze the value of the discriminant of a quadratic equation will usually require us to solve a quadratic inequality, as the next example illustrates.

Example 35 – A quadratic from evaluating a discriminant

Given $f(x) = 3kx^2 - (k+3)x + k - 2$, find the range of values of k for which f has no real zeros.

Solution

The quadratic function f will have no real zeros when its discriminant is negative. Since f is written in the form $ax^2 + bx + c = 0$ then, in terms of the parameter k, $a = 3k$, $b = -(k+3)$ and $c = k - 2$. Substituting these values into the discriminant, we have the inequality

$(-(k+3))^2 - 4(3k)(k-2) < 0$

$k^2 + 6k + 9 - 12k^2 + 24k < 0$

$-11k^2 + 30k + 9 < 0$ Easier to factorize if leading coefficient is positive.

$11k^2 - 30k - 9 > 0$ Multiply both sides by -1; reverse inequality sign.

$$k = \frac{-(-30) \pm \sqrt{(-30)^2 - 4(11)(-9)}}{2(11)} = \frac{30 \pm \sqrt{1296}}{22} = \frac{30 \pm 36}{22}$$

$$k = \frac{30+36}{22} = \frac{66}{22} = 3 \text{ or } k = \frac{30-36}{22} = -\frac{6}{22} = -\frac{3}{11}$$

The two rational zeros indicate $11k^2 - 30k - 9$ could have been factorized into $(11k+3)(k-3)$:

$$(11k+3)(k-3) > 0$$

The results of the sign chart indicate that the solution set to the inequality is $k < -\frac{3}{11}$ or $k > 3$. Therefore, any value of k such that $k < -\frac{3}{11}$ or $k > 3$ will cause the function f to have no real zeros.

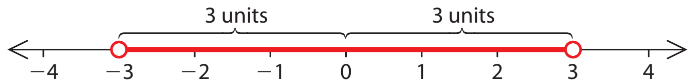
sign chart

k		$-\frac{3}{11}$		3	
$11k+3$	$-$	0	$+$		$+$
$k-3$	$-$		$-$	0	$+$
$(11k+3)(k-3)$	$+$	0	$-$	0	$+$

Absolute value inequalities

In Section 1.1 we described how absolute value is used to indicate distance on the number line. For example, the equation $|x| = 3$ means that some number x is a distance of 3 units from the origin. The two solutions to this equation are $x = 3$ and $x = -3$. Consequently, the inequality $|x| < 3$ means that x lies *at most* 3 units from the origin, as shown in Figure 3.13.

Figure 3.13



This means that x lies *between* -3 and 3, that is, $-3 < x < 3$. Similarly, the inequality $|x| > 3$ means that x lies 3 *or more* units from the origin. This occurs if x is to the left of -3 (that is, $x < -3$) or if x lies to the right of 3 (that is, $x > 3$).

Properties of absolute value inequalities

For any real numbers x and c such that $c > 0$:

1. $|x| < c$ if and only if $-c < x < c$.
2. $|x| > c$ if and only if $x < -c$ or $x > c$.

Example 36 – Absolute value inequality I

Solve for x: $|3x - 7| \geqslant 8$

Solution

Applying the second property for absolute value inequalities, we have

$$3x - 7 \leqslant -8 \text{ or } 3x - 7 \geqslant 8$$
$$3x \leqslant -1 \text{ or } 3x \geqslant 15$$
$$x \leqslant -\tfrac{1}{3} \text{ or } x \geqslant 5$$

Therefore, the solution set is the union of two half-open intervals $x \leqslant -\frac{1}{3}$ or $x \geqslant 5$, which can also be written in interval notation as $]-\infty, -\frac{1}{3}] \cup [5, \infty[$.

Example 37 – Absolute value inequality II

Find the values of x which satisfy the inequality $\left|\dfrac{x}{x+4}\right| < 2$.

Solution

Applying the first property for absolute value inequalities gives

$$-2 < \frac{x}{x+4} < 2$$

We cannot multiply both sides by $x + 4$ unless we take into account the two different cases: (1) when $x + 4$ is positive (inequality is *not* reversed), and (2) when $x + 4$ is negative (inequality sign *is* reversed). Instead, let's solve the two inequalities in the 'combined' inequality separately by rearranging so that zero is on one side and then analyze where the expression on the other side is zero, positive and negative. This is similar to the approach used in Example 34.

$\frac{x}{x+4} > -2 \quad$ and $\quad \frac{x}{x+4} < 2\;$ the word ‘and’ indicates intersection

$\frac{x}{x+4} + 2 > 0 \quad$ and $\quad \frac{x}{x+4} - 2 < 0$

$\frac{x}{x+4} + \frac{2x+8}{x+4} > 0 \quad$ and $\quad \frac{x}{x+4} - \frac{2x+8}{x+4} < 0$

$\frac{3x+8}{x+4} > 0 \quad$ and $\quad \frac{-x-8}{x+4} < 0$

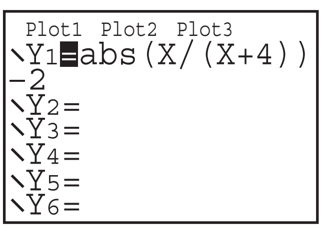
x		-4		$-\frac{8}{3}$	
$3x+8$	$-$		$-$	0	$+$
$x+4$	$-$	0	$+$		$+$
$\frac{3x+8}{x+4}$	$+$	X	$-$	0	$+$

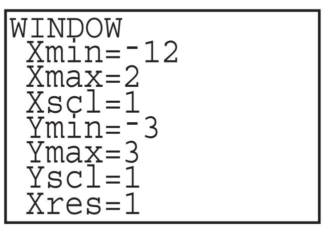
x		-8		-4	
$-x-8$	$+$	0	$-$		$-$
$x+4$	$-$		$-$	0	$+$
$\frac{-x-8}{x+4}$	$-$	0	$+$	X	$-$

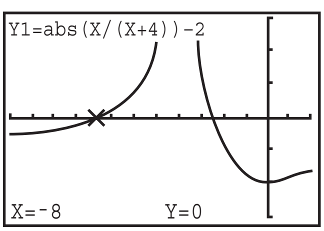
$x < -4 \cup x > -\frac{8}{3} \qquad \cap \qquad x < -8 \cup x > -4$

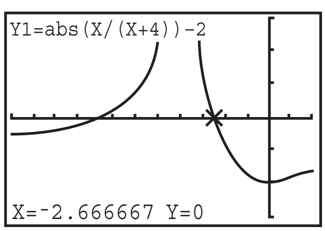
The solution set for the original ‘combined’ inequality, $-2 < \frac{x}{x+4} < 2$, will be the intersection of the solution sets of the two separate inequalities graphed above on the number line. Thus, the solution set is $x < -8$ or $x > -\frac{8}{3}$.

A graphical check using a GDC can be effectively performed by graphing the equation $y = \left|\frac{x}{x+4}\right| - 2$ and observing where the graph is below the x-axis. The values of x for which this is true will correspond to the solution set for the inequality $\left|\frac{x}{x+4}\right| < 2$.









Example 38 – Algebraic and graphical methods

Solve the inequality $|x - 4| > 2|x - 7|$.

Solution

Method 1 – Algebraic

If $a > 0$, $b > 0$ and $a = b$, then $a^2 = b^2$. Since the expressions on both sides must be positive then we can square both sides and remove the absolute value signs.

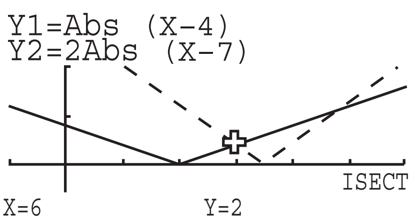
$$(x - 4)^2 > (2(x - 7))^2$$
$$x^2 - 8x + 16 > 4(x^2 - 14x + 49)$$
$$x^2 - 8x + 16 > 4x^2 - 56x + 196$$
$$0 > 3x^2 - 48x + 180$$
$$0 > x^2 - 16x + 60$$
$$(x - 10)(x - 6) < 0$$

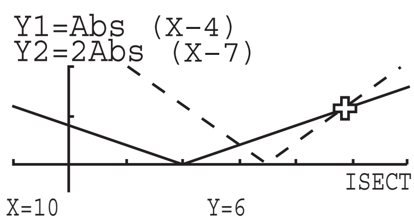
x		6		10	
$x - 10$	$-$		$-$	0	$+$
$x - 6$	$-$	0	$+$		$+$
$(x - 10)(x - 6)$	$+$	0	$-$	0	$+$

Therefore, the solution set is the open interval $6 < x < 10$.

Method 2 - Graphical

We can graph the two equations $y_1 = |x - 4|$ and $y_2 = 2|x - 7|$ and use our GDC to determine for what values of x the graph of y_1 is above the graph of y_2.





The equation $y_2 = 2|x - 7|$ has been graphed in a dashed style. By using the 'intersect' command on the GDC we find that the graph of y_1 is above the graph of y_2 for $6 < x < 10$. Therefore, the solution set is the open interval $6 < x < 10$.

Example 39 – Inequality involving rational expressions

For what values of x is $\frac{x}{x + 8} \leqslant \frac{1}{x - 1}$? Solve algebraically.

Solution

As applied in previous examples, an effective algebraic approach is to rearrange the inequality so that both fractions are on the same side with zero on the other side. Then combine the two fractions into one fraction and analyze where the fraction is zero, positive and negative.

x		-8		-2		1		4	
$x+2$	$-$		$-$	0	$+$		$+$		$+$
$x-4$	$-$		$-$		$-$		$-$	0	$+$
$x+8$	$-$	0	$+$		$+$		$+$		$+$
$x-1$	$-$		$-$		$-$	0	$+$		$+$
$\frac{(x+2)(x-4)}{(x+8)(x-1)}$	$+$	X	$-$		$+$	X	$-$		$+$

$$\frac{x}{x+8} - \frac{1}{x-1} \leqslant 0$$

$$\frac{x(x-1)-(x+8)}{(x+8)(x-1)} \leqslant 0$$

$$\frac{x^2-2x-8}{(x+8)(x-1)} \leqslant 0$$

$$\frac{(x+2)(x-4)}{(x+8)(x-1)} \leqslant 0$$

Therefore, $\frac{x}{x+8} \leqslant \frac{1}{x-1}$ when $-8 < x \leqslant -2$ or $1 < x \leqslant 4$, which can also be expressed in interval notation as $]-8, -2] \cup]1, 4]$.

Exercise 3.5

In questions 1–22, solve for x in the equation. If possible, find all real solutions and express them exactly. If this is not possible, then solve using your GDC and approximate any solutions to three significant figures. Be sure to check answers and to recognize any extraneous solutions.

1 $\sqrt{x+6} + 2x = 9$

2 $\sqrt{x+7} + 5 = x$

3 $\sqrt{7x+14} - 2 = x$

4 $\sqrt{2x+3} - \sqrt{x-2} = 2$

5 $\frac{5}{x+4} - \frac{4}{x} = \frac{21}{5x+20}$

6 $\frac{x+1}{2x+3} = \frac{5x-1}{7x+3}$

7 $\frac{1}{x} - \frac{1}{x+1} = \frac{1}{x+4}$

8 $\frac{2x}{1-x^2} + \frac{1}{x+1} = 2$

9 $x^4 - 2x^2 - 15 = 0$

10 $2x^{\frac{2}{3}} - x^{\frac{1}{3}} - 15 = 0$

11 $x^6 - 35x^3 + 216 = 0$

12 $5x^{-2} - x^{-1} - 2 = 0$

13 $|3x+4| = 8$

14 $|x+6| = |3x-24|$

15 $|5x+1| = 2x$

16 $|x-1| + |x| = 3$

17 $\left|\frac{x+1}{x-1}\right| = 3$

18 $\sqrt{x} - \frac{6}{\sqrt{x}} = 1$

19 $\sqrt{4-x} - \sqrt{6+x} = \sqrt{14+2x}$

20 $\frac{6}{x^2+1} = \frac{1}{x^2} + \frac{10}{x^2+4}$

21 $x - \sqrt{x+10} = 0$

22 $6x - 37\sqrt{x} + 56 = 0$

In questions 23–30, find the values of x that solve the inequality.

23 $3x^2 - 4 < 4x$

24 $\frac{2x-1}{x+2} \geqslant 1$

25 $2x^2 + 8x \leqslant 120$

26 $|1-4x| > 7$

27 $|x-3| > |x-14|$

28 $\left|\frac{x^2-4}{x}\right| \leqslant 3$

29 $\frac{x}{x-2} > \frac{1}{x+1}$

30 $\frac{4x-1}{x^2-2x-3} < 3$

31 Find the values of p for which the equation $px^2 - 3x + 1 = 0$ has a) one real solution, b) two real solutions, and c) no real solutions.

32 Given $f(x) = x^2 + x(k-1) + k^2$, find the range of values of k so that $f(x) > 0$ for all real values of x.

33 Show that both of the following inequalities are true for all real numbers m and n such that $m > n > 0$.

a) $m + \frac{1}{n} > 2$ b) $(m+n)\left(\frac{1}{m} + \frac{1}{n}\right) > 4$

34 Find all of the exact solutions to the equation $(x^2 + x)^2 = 5x^2 + 5x - 6$.

35 If a, b and c are positive and unequal, show that $(a + b + c)^2 < 3(a^2 + b^2 + c^2)$.

36 Find the values of x that solve each inequality.

a) $\left|\frac{2x-3}{x}\right| < 1$ b) $\frac{3}{x-1} - \frac{2}{x+1} < 1$

37 Provide a geometric or algebraic argument to show that $|a + b| \leqslant |a| + |b|$ for all $a, b \in \mathbb{R}$.

3.6 Partial fractions (Optional)

In arithmetic, when we add fractions we find the least common denominator. Then we multiply both the numerator and denominator of each term by what is needed to complete the common denominator. For example:

$$\frac{2}{3} + \frac{5}{7} = \frac{2}{3} \cdot \frac{7}{7} + \frac{5}{7} \cdot \frac{3}{3} = \frac{14 + 15}{21} = \frac{29}{21}$$

$$\frac{2}{3} + \frac{5}{9} + \frac{1}{27} = \frac{2}{3} \cdot \frac{9}{9} + \frac{5}{9} \cdot \frac{3}{3} + \frac{1}{27} = \frac{18 + 15 + 1}{27} = \frac{34}{27}$$

Reversing the process is called expressing each compound fraction as *partial fractions*. That is, given for example the fraction $\frac{29}{21} = \frac{29}{3 \times 7}$, we express it as a sum of two fractions. One fraction has denominator 3 and the other has denominator 7. Hence, we have the name *partial fractions*.

The process of finding the *partial fractions* is a straightforward process. We write:

$\frac{29}{3 \times 7} = \frac{a}{3} + \frac{b}{7}$ and then we solve for two integers a and b.

$$\frac{29}{3 \times 7} = \frac{a}{3} + \frac{b}{7} = \frac{7a + 3b}{21} \Rightarrow 7a + 3b = 29$$

Now by trial and error we can find that $a = 2$ and $b = 5$. Other answers are also possible $(-1, 12)$, $(8, -9)$ …

Notice the situation in the second example. The L.C.M. contains different powers of the same number. Consequently, when finding the partial fractions decomposition you need to consider that all powers less than or equal to the highest one may be present. That is, when we set up the process of decomposing $\frac{24}{27}$ we set it up in the following manner:

$$\frac{24}{27} = \frac{a}{27} + \frac{b}{9} + \frac{c}{3}$$

Then we attempt to find the values of a, b, and c.

In algebra, we carry out that process on the addition of rational expressions. Once again we multiply the numerator and denominator of each term by what was missing from the denominator of that term.

Partial fractions decomposition (PFD)

With partial fractions decomposition, we are going to reverse the process and decompose a rational expression into two or more simpler proper rational expressions. This is a very useful skill in which a single fraction with a factorable denominator is split into the sum of two or more fractions (partial fractions) whose denominators are the factors of the original denominator.

For example: $\frac{12x - 1}{2x^2 - 5x - 3} = \frac{2}{2x + 1} + \frac{5}{x - 3}$

The method of partial fractions decomposition is extremely helpful in evaluating certain integrals as you will see in Section 16.5 (optional).

Example 40

Find the partial fraction decomposition of $\frac{x + 1}{x^2 + 5x + 6}$.

Solution

$\frac{x + 1}{x^2 + 5x + 6} \equiv \frac{x + 1}{(x + 2)(x + 3)}$, and hence we will attempt to find two numbers a and b such that:

$$\frac{x + 1}{x^2 + 5x + 6} \equiv \frac{a}{x + 2} + \frac{b}{x + 3}$$

(Notice that we wrote this as an identity rather than equality because it has to be true for all values of x and not only for a few.)

$$\frac{x + 1}{x^2 + 5x + 6} \equiv \frac{a}{x + 2} + \frac{b}{x + 3} \equiv \frac{a(x + 3) + b(x + 2)}{(x + 2)(x + 3)}$$

Since the denominators of these identical fractions are the same, their numerators must also be the same. That is

$$x + 1 \equiv a(x + 3) + b(x + 2).$$

We have two methods of solution here.

First method

$$x + 1 \equiv a(x + 3) + b(x + 2) \Leftrightarrow x + 1 \equiv (a + b)x + (3a + 2b)$$

For two polynomials to be identical, the coefficients of the same powers must be the same, that is, the coefficient of x on the left must be the same as the coefficient of x on the right and similarly the constant terms. Hence:

$1 = a + b$ and $1 = 3a + 2b$

Now, solving the system with two equations will yield:

$a = -1$ and $b = 2$

Hence, $\frac{x+1}{x^2+5x+6} \equiv \frac{-1}{x+2} + \frac{2}{x+3}$.

This is also called the 'cover-up' method. This method allows the choice of numbers that are not initially in the domain of the original rational expression.

Second method

$x + 1 \equiv a(x+3) + b(x+2)$

Again, since this is an identity, the two sides must be the same for any choice of x. Hence, we can substitute any two numbers for x to get the value of each of a and b, specifically replacing x with -3 yields:

$x + 1 \equiv a(x+3) + b(x+2) \Rightarrow -2 = -b \Rightarrow b = 2.$

Notice how the choice of -3 eliminated the term with a and allowed us to find b directly. Replacing x with -2 yields:

$x + 1 \equiv a(x+3) + b(x+2) \Rightarrow -1 = a.$

This is of course the same result as above. Also notice here how the choice of -2 eliminated the term with b and allowed us to find a directly.

Note: This method is helpful in cases where there are no repeated factors.

The second method is faster whenever applicable. (We will discuss this in more detail later.)

Example 41

Find the PFD for $\frac{5x^2+16x+17}{2x^3+9x^2+7x-6}$.

Solution

$$\frac{5x^2+16x+17}{2x^3+9x^2+7x-6} \equiv \frac{5x^2+16x+17}{(2x-1)(x+2)(x+3)} \equiv \frac{a}{2x-1} + \frac{b}{x+2} + \frac{c}{x+3}$$

First method

$$5x^2 + 16x + 17 \equiv a(x+2)(x+3) + b(2x-1)(x+3) + c(2x-1)(x+2)$$
$$\equiv (a+2b+2c)x^2 + (5a+5b+3c)x + 6a - 3b - 2c$$

This leads to this system: $\begin{cases} a+2b+2c=5 \\ 5a+5b+3c=16 \\ 6a-3b-2c=17 \end{cases}$

Using any method of your choice for solving systems of equations, you should have:

$a = 3, b = -1, c = 2$ and hence:

$$\frac{5x^2+16x+17}{2x^3+9x^2+7x-6} \equiv \frac{3}{2x-1} - \frac{1}{x+2} + \frac{2}{x+3}$$

Second method

$5x^2 + 16x + 17 \equiv a(x + 2)(x + 3) + b(2x - 1)(x + 3) + c(2x - 1)(x + 2)$

$x = -2 \Rightarrow 5 = -5b \Rightarrow b = -1$

$x = -3 \Rightarrow 14 = 7c \Rightarrow c = 2$

$x = \frac{1}{2} \Rightarrow \frac{105}{4} = \frac{35}{4}a \Rightarrow a = 3$

Properties

1 Partial fractions decomposition only works for proper rational expressions, that is, the degree of the numerator must be less than the degree of the denominator. If it is not, then you must perform long division first, and then perform the partial fractions decomposition on the rational part (the remainder over the divisor). After you've done the partial fraction decomposition, just add back in the quotient part from the long division.

2 **Linear factors**: We can only decompose the partial fractions into proper rational expressions. Hence, in each partial fraction, when the denominator is linear, only a constant can be in the numerator. So, for every linear factor in the denominator, you will need a constant in the numerator. See Examples 40 and 41 above.

3 **Repeated linear factors**: If the denominator of the rational expression contains repeated linear factors, then following our discussion in the introduction, the process is as follows.

We need to include a factor in the expansion for each power possible. For example, if we have $(x - 1)^3$, we will need to include $(x - 1)$, an $(x - 1)^2$, and $(x - 1)^3$. Each of those $(x - 1)$ factors would have a constant term in the numerator because $x - 1$ is linear, no matter what power it is raised to.

For example: $\frac{13x^3 - 62x^2 + 101x - 58}{(x - 1)^3(2x - 5)} \equiv \frac{a}{(x - 1)^3} + \frac{b}{(x - 1)^2} + \frac{c}{x - 1} + \frac{d}{2x - 5}$

4 **Irreducible quadratic factors:** If the rational expression we are decomposing contains irreducible quadratic factors in the denominator, then the numerator could have a linear term and/or a constant term. So, for every irreducible quadratic factor in the denominator, you will need a linear term and a constant term in the numerator.

For example: $\frac{-8x^3 + 15x^2 - 26x + 33}{(x - 1)^2(2x^2 + 5)} \equiv \frac{a}{(x - 1)^2} + \frac{b}{x - 1} + \frac{cx + d}{2x^2 + 5}$

Note: It may turn out that any of the numbers a, b, c, or d is zero.

Example 42

Write $\frac{3x - 1}{x^2 + 4x + 4}$ as the sum of partial fractions.

Solution

The first step is to factorise the denominator.

$x^2 + 4x + 4 = (x + 2)^2$

Here the denominator has a repeated linear factor: $\frac{3x - 1}{x^2 + 4x + 4} = \frac{3x - 1}{(x + 2)^2}$

Because there are two (i.e. repeated) linear factors of $x + 2$ in the denominator of the original rational expression then it *must* have a partial fraction with a denominator of $(x + 2)^2$, and it *may* also have a partial fraction with a denominator of $x + 2$.

Thus, we are looking for constants A and B such that:

$$\frac{3x - 1}{(x + 2)^2} \equiv \frac{A}{x + 2} + \frac{B}{(x + 2)^2}$$

Multiplying both sides of the equation by $(x + 2)^2$ gives:

$$3x - 1 \equiv A(x + 2) + B$$

Essentially, the task is to find the unique values of A and B such that this equation is an identity, i.e. it is true for all values of x for which the original fraction is defined (in this case $x \neq -2$). However, as you recall, the 'cover-up' method allows us to choose 'helpful' values of x including such numbers. For example, in this case, if $x = -2$ then A is eliminated and the value of B can be found directly.

Let $x = -2$: $3x - 1 \equiv A(x + 2) + B \Rightarrow 3(-2) - 1 = A \cdot 0 + B$
$\Rightarrow B = -7$

Let $x = 0$: $3x - 1 \equiv A(x + 2) + B \Rightarrow 3 \cdot 0 - 1 = 2A - 7$
$\Rightarrow 2A = 6$
$\Rightarrow \mathrm{A} = 3$

Therefore, $\dfrac{3x - 1}{x^2 + 4x + 4} = \dfrac{3}{x + 2} - \dfrac{7}{(x + 2)^2}$

Example 43

Write $\dfrac{2}{x^3 + 3x^2 + 2x}$ as the sum of partial fractions.

Solution

We first factorize the denominator and discover that one of the factors is an irreducible quadratic factor:

$$\frac{2}{x^3 + 3x^2 + 2x} = \frac{2}{x(x^2 + 2x + 2)} \equiv \frac{a}{x} + \frac{bx + c}{x^2 + 2x + 2}$$

Simplifying the expression gives:

$$2 \equiv a(x^2 + 2x + 2) + x(bx + c) \Rightarrow 2 \equiv (a + b)x^2 + (2a + c)x + 2a \Rightarrow$$

$$\begin{cases} a + b = 0 \\ 2a + c = 0 \\ 2a = 2 \end{cases} \Rightarrow \begin{cases} a = 1 \\ b = -1 \\ c = -2 \end{cases}$$

Therefore $\dfrac{2}{x^3 + 2x^2 + 2x} = \dfrac{1}{x} - \dfrac{x + 2}{x^2 + 2x + 2}$.

Exercise 3.6

Decompose each of the following rational expressions into partial fractions.

1 $\dfrac{5x+1}{x^2+x-2}$

2 $\dfrac{x+4}{x^2-2x}$

3 $\dfrac{x+2}{x^2+4x+3}$

4 $\dfrac{5x^2+20x+6}{x^3+2x^2+x}$

5 $\dfrac{2x^2+x-12}{x^3+5x^2+6x}$

6 $\dfrac{4x^2+2x-1}{x^3+x^2}$

7 $\dfrac{3}{x^2+x-2}$

8 $\dfrac{5-x}{2x^2+x-1}$

9 $\dfrac{3x+4}{(x+2)^2}$

10 $\dfrac{12}{x^4-x^3-2x^2}$

11 $\dfrac{2}{x^3+x}$

12 $\dfrac{x+2}{x^3+3x}$

13 $\dfrac{3x+2}{x^3+6x}$

14 $\dfrac{2x+3}{x^3+8x}$

15 $\dfrac{x+5}{x^3-4x^2-5x}$

Practice questions

1 Solve for x in the equation $x^2-(a+3b)x+3ab=0$.

2 Find the values of x that solve the following inequality.

$$\frac{3x-2}{5}+3 \geqslant \frac{4x-1}{3}$$

3 For what value of c is the vertex of the parabola $y=3x^2-8x+c$ at $\left(\frac{4}{3}, -\frac{1}{3}\right)$?

4 The quadratic function $f(x)=ax^2+bx+c$ has the following characteristics:

(i) passes through the point (2, 4); **(ii)** has a maximum value of 6 when $x=4$; and **(iii)** has a zero of $x=4+2\sqrt{3}$

Find the values of a, b and c.

5 If the roots of the equation $x^3+5x^2+px+q=0$ are ω, 2ω and $\omega+3$, find the values of ω, p and q.

6 Find all values of m such that the equation $mx^2-2(m+2)x+m+2=0$ has **a)** two real roots; **b)** two real roots (one positive and one negative).

7 $x-1$ and $x+1$ are factors of the polynomial x^3+ax^2+bx+c, and the polynomial has a remainder of 12 when divided by $x-2$. Find the values of a, b and c.

8 Solve the inequality $|x|<5|x-6|$.

9 Find the range of values for k in order for the equation $2x^2+(3-k)x+k+3=0$ to have two imaginary solutions.

10 Consider the rational function $f(x)=\dfrac{2x^2+8x+7}{x^2+4x+5}$. Do not use your GDC for this question.

a) Write $f(x)$ in the form $a-\dfrac{b}{(x+c)^2+d}$.

b) State the values of **(i)** $\lim_{x\to+\infty} f(x)$, and **(ii)** $\lim_{x\to-\infty} f(x)$.

c) State the coordinates of the minimum point on the graph of $f(x)$.

11 Find the values of k so that the equation $(k - 2)x^2 + 4x - 2k + 1 = 0$ has two distinct real roots.

12 When the function $f(x) = 6x^4 + 11x^3 - 22x^2 + ax + 6$ is divided by $(x + 1)$ the remainder is -20. Find the value of a.

13 The polynomial $p(x) = (ax + b)^3$ leaves a remainder of -1 when divided by $(x + 1)$, and a remainder of 27 when divided by $(x - 2)$. Find the values of the real numbers a and b.

14 The polynomial $f(x) = x^3 + 3x^2 + ax + b$ leaves the same remainder when divided by $(x - 2)$ as when divided by $(x + 1)$. Find the value of a.

15 When the polynomial $x^4 + ax + 3$ is divided by $(x - 1)$, the remainder is 8. Find the value of a.

16 The polynomial $x^3 + ax^2 - 3x + b$ is divisible by $(x - 2)$ and has a remainder 6 when divided by $(x + 1)$. Find the value of a and of b.

17 The polynomial $x^2 - 4x + 3$ is a factor of $x^3 + (a - 4)x^2 + (3 - 4a)x + 3$. Calculate the value of the constant a.

18 Consider $f(x) = x^3 - 2x^2 - 5x + k$. Find the value of k if $(x + 2)$ is a factor of $f(x)$.

19 Find the real number k for which $1 + ki$ $(i = \sqrt{-1})$ is a zero of the polynomial $z^2 + kz + 5$.

20 The equation $kx^2 - 3x + (k + 2) = 0$ has two distinct real roots. Find the set of possible values of k.

21 Consider the equation $(1 + 2k)x^2 - 10x + k - 2 = 0$, $k \in \mathbb{R}$. Find the set of values of k for which the equation has real roots.

22 Find the range of values of m such that for all x

$$m(x + 1) \leqslant x^2.$$

23 Find the values of x for which $|5 - 3x| \leqslant |x + 1|$.

24 Solve the inequality $x^2 - 4 + \dfrac{3}{x} < 0$.

25 Solve the inequality $|x - 2| \geqslant |2x + 1|$.

26 Let $f(x) = \dfrac{x + 4}{x + 1}$, $x \neq -1$ and $g(x) = \dfrac{x - 2}{x - 4}$, $x \neq 4$.

Find the set of values of x such that $f(x) \leqslant g(x)$.

27 Solve the inequality $\left|\dfrac{x + 9}{x - 9}\right| \leqslant 2$.

28 Given that $2 + i$ is a root of the equation $x^3 - 6x^2 + 13x - 10 = 0$ find the other two roots.

29 Find all values of x that satisify the inequality $\dfrac{2x}{|x - 1|} < 1$.